A Quasi-Static Method for Solving Transient Problems in Weakly Conducting Media

Th. Heubrandtner *[1], B. Schnizer[1] und L. Dědek[2]

[1]Institut für Theoretische Physik, Technische Universität Graz, A-8010 Graz; Email: Schnizer@itp.tu-graz.ac.at
[2]Faculty of Electr. Eng. and Comp. Science, Purkynova 118, CZ-612 00 Brno

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Th. Heubrandtner [a1], B. Schnizer[1] und L. Dědek[2]
[1]Institut für Theoretische Physik, Technische Universität Graz, A-8010 Graz; Email: Schnizer@itp.tu-graz.ac.at
[2]Faculty of Electr. Eng. and Comp. Science, Purkynova 118, CZ-612 00 Brno

[a]Research supported by Steiermärkische Landesregierung, Graz.

Zusammenfassung. Transiente elektrodynamische Probleme sind bekannt schwierig zu lösen. Doch für viele Probleme gibt es quasi-statische Näherungen (Fano, Chu und Adler (1968), Haus und Melcher (1989)) die es gestatten, derartige Probleme für schwach leitende Medien viel einfacher als bisher zu lösen. Das zeitabhängige elektrische Feld kann mittels eines skalaren Potentials (statt eines Vektorpotentials oder eines Greenschen Tensors) berechnet werden, das einer partiellen Differentialgleichung 3. Ordnung, die in manchem den Charakter einer Diffusionsgleichung hat, gehorcht. Diese Methode wird an einigen Problemen in der Wechselstromtechnik und in der Theorie der Zähler vorgestellt. Dies sind Ein- oder Zweischichtenprobleme; für letztere wird das Problem der Selbstadjungiertheit des Operators besonders untersucht. Damit werden Probleme mit zeitlich anwachsenden Ladungen oder mit gleichförmig bewegten Ladungen gelöst. Einige dieser neuen Näherungslösungen wurden mit solchen verglichen, die mittels eines Vektorpotentials berechnet wurden. Auf diese Weise wurde den Gültigkeitsbereich der neuen Näherung überprüft (Schöpf and Schnizer (1992), Heubrandter (1999)) . Dieser beträgt für die Leitfähigkeit σ im Bereich von $0 \leq \sigma \leq 10^{-3}\ S/m$.

Abstract. Transient electrodynamical problems are notoriously difficult to solve. For many such problems a quasi-static approximate theory for materials with low conductivity (Fano, Chu und Adler (1968), Haus und Melcher (1989)) is available permitting a much simpler solution. The time-dependent electric field may be derived from a scalar potential (in place from a vector potential or a Green's tensor), which is the solution of a third order partial differential equation, whose type corresponds to that of a diffusion equation. This method is presented for some problems of alternating currents and of particle counters. These are problems with one or two layers. The question of self-adjoint operators is very important for the latter and is investigated. Problems with increasing or moving charges are solved. Some of these quasi-static solutions have been compared to ones obtained by a more rigorous approach using a vector potential for time-dependent electromagnetic fields (Schöpf and Schnizer (1992), Heubrandter (1999)). Agreement is found for conductivities in the range $0 \leq \sigma \leq 10^{-3}\ S/m$.

1 Introduction

Transient electrodynamical problems are notoriously difficult to solve. In the general case, one needs Green's tensors or a scalar and a vector potential. The retardation contained in the equations also brings complications. Solutions depending on roots are involved so that branch cuts must be taken into account. In media with a conductivity a static field is impossible. The pertinent relaxation time is given by Stratton (1941) as

$$\tau_R = \varepsilon_0\varepsilon/\sigma. \quad (1)$$

ε is the relative dielectric constant, σ the conductivity of the medium. This rate is very short for metals. For example, for copper σ is of the order of 10^7 and τ_R is as short as $10^{-18}\ s$. But nowadays materials with a much lower conductivity are used. For example, in Resistive Plate Chambers, a new type of particle counters under developement at the European Laboratory for Particle Physics (CERN), Geneva, the name-giving plates (Crotty et al., 1995) have a relative dielectric constant with a value of 2 - 4 and a conductivity of about $10^{-9}\ S/m$. Then the decay rate $\tau_R \approx 10^{-3}\ s$ is long as compared to the other times constants as, for example, the time the electromagnetic field needs to transverse the panel, $10^{-10}\ s$, or the electron cloud produced by the primary particle to be counted needs to transverse the chamber, $10^{-7}\ s$. So this is field where a quasi-static theory could be applied. In fact, such a theory is available and was developed systematically by several people working at the MIT (Fano, Chu und Adler (1963), Haus und Melcher (1989)). It was implemented in codes for numerical field calculations by Dědek and Bachorec (1998) and Dědek (1999).

In this paper the derivation of the fundamental equation is reported and some simple applications are shown. For a single medium it is shown how the admittance of the time-harmonic problem is connected with the capacitance of the static problem. If two media with different electric properties are present, the question of the self-adjoint character of the problem is imposing due to the continuity conditions at the interface. This is solved by defining an appropriate inner product. The method so developed is used to compute some fields induced by time-dependent charges. In particular, it is shown that the field due to a charge newly created in an isolator at first penetrates the adjacent weakly conducting medium but is thereafter destroyed with the relaxation time τ_R defined in eq.(1).

2 Derivation of the Quasi-Static Approximation

This derivation starts from the complete Maxwell equations

$$\nabla \times \vec{E} \;=\; \frac{\partial \vec{B}}{\partial t} \, , \tag{2}$$

$$\nabla \times \vec{H} \;=\; \vec{j} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon_0 \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{j}_e \, ; \tag{3}$$

$$\nabla \cdot \vec{B} \;=\; 0 \, , \qquad \nabla \cdot \vec{D} = \nabla \cdot (\varepsilon_0 \varepsilon \vec{E}) = \rho \, . \tag{4}$$

It is assumed that there is a weak current density due to the electric field and an impressed current density j_e. The quasi-static assumption is that the sources act slowly, so that the fields change slowly, and that the conductivity σ is rather small. Therefore the magnetic field, thus the solenoidal part of the electric field, are negligibly small. For that reason the electric field may be derived from a scalar potential:

$$\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \approx 0 \;\; \Rightarrow \;\; \vec{E} = -\nabla \Phi \, . \tag{5}$$

The last relation is inserted into the equation resulting from taking the divergence of the second Maxwell equation (3). This yields the "Poisson" equation for a time-dependent impressed current or charge density, which are assumed to fulfil a continuity condition.

$$\boxed{\sigma \nabla^2 \Phi + \varepsilon_0 \varepsilon \, \frac{\partial}{\partial t} \nabla^2 \Phi = -\frac{\partial \rho_e}{\partial t} \; = div \; \vec{j}_e .} \tag{6}$$

Frequently the above equation is solved by a Fourier or Laplace transform. For the latter we get:

$$\bar{\Phi}(\vec{r}, s) \;=\; \mathcal{L}\big[\Phi(r,t)\big] \;:=\; \int_0^\infty \; e^{-st} \; \Phi(\vec{r}, t) \, dt. \tag{7}$$

From this it follows:

$$\mathcal{L}\Big[\frac{\partial}{\partial t} \Phi(r,t)\Big] \;:= s \, \bar{\Phi}(\vec{r}, s) - \Phi(\vec{r}, t=0); \qquad \mathcal{L}\Big[\frac{\partial}{\partial t} \rho(r,t)\Big] \;:= s \, \bar{\rho}(\vec{r}, s) - \rho(\vec{r}, t=0). \tag{8}$$

$\Phi(\vec{r}, t=0)$ and $\rho(\vec{r}, t=0)$ are the initial data of the potential and of the charge. If both are zero we get in place of eq.(6) the follwing one:

$$\nabla^2 \bar{\Phi}(\vec{r}, s) \;=\; -\frac{\bar{\rho}(\vec{r}, s)}{\varepsilon_0 \epsilon} \quad \text{with} \quad \epsilon := \varepsilon + \sigma/(s \varepsilon_0). \tag{9}$$

In the quasi-static approximation we get an equation which resembles a Poisson equation in place of a Helmholtz equation. This is very advantangeous as the toolbox for solving the former equation is much richer than that for solving the latter.

3 One-Layer Problems

At first some problems are solved, in which a homogeneous weakly conducting dielectric fills all space between the electrodes. The homogeneous equation belonging to eq.(6) may be solved by separating the time dependence from that on the space coordinates: $\Phi(\vec{r}, t) \;:=\; \Psi(\vec{r}) \; T(t)$. This gives the following differential equation for the time function $T(t)$, whose solution contains the relaxation time τ_R of eq.(1):

$$\varepsilon_0 \varepsilon \, \frac{dT}{dt} \;+\; \sigma \, T \;=\; 0, \qquad T = T_0 \; e^{-t/\tau_R}; \qquad \Phi(\vec{r}, t) := \Psi(\vec{r}) \; e^{-t/\tau_R}. \tag{10}$$

It is interesting to note that the time behaviour of the solution, i.e. the decay of the field in the weakly conducting medium, is completely determined by this time factor. The geometry of the electrodes has no influence. On the other hand, for an impressed charge density with a harmonic time dependence $e^{j\omega t}$ one gets in place of eq.(9) the following equation with the corresponding stationary solution:

$$\nabla^2\Psi(\vec{r}) \;=\; -\frac{\rho(\vec{r})}{\varepsilon_0\epsilon} \quad \text{with} \quad \epsilon := \varepsilon + \sigma/(j\omega\varepsilon_0). \tag{11}$$

3.1 The Spherical Condensor

A spherical condensor consists of two concentric spherical eletrodes of radius $r = a,\ b > a$ respectively. At the starting time $t = 0$ the electrodes have charges $\pm Q_0$. These decay due to the losses in the dielectric. One may assume that the outer electrode is grounded, so that it is at potential zero. The corresponding solution is:

$$\Phi(r,t) \;=\; \frac{Q_0}{4\pi\varepsilon_0\varepsilon}\left(\frac{1}{r} \;-\; \frac{1}{b}\right) e^{-t/\tau_R}; \quad Q(t) = Q_0\; e^{-t/\tau_R};\;\; V(t) = V_0\; e^{-t/\tau_R}; \quad V_0 \;=\; \frac{Q_0}{4\pi\varepsilon_0\varepsilon}\,\frac{ab}{b-a}. \tag{12}$$

The charge and the voltage decay with the same decay rate τ_R defined in eq.(1). The exponential decay law is independent of the geometry. For the current flowing one may write (with $r = a$ or b) :

$$I \;=\; 4\pi r^2 j_r \;=\; 4\pi\sigma r^2 E_r \;=\; -4\pi\sigma r^2\,\frac{\partial\Phi}{\partial r} \;=\; \frac{Q_0\sigma}{\varepsilon_0\varepsilon}\; e^{-t/\tau_R} = G\; V_0\; e^{-t/\tau_R}. \qquad G = \frac{C\sigma}{\varepsilon_0\varepsilon} = 4\pi\,\frac{ab\;\sigma}{b-a}. \tag{13}$$

So one concludes there is a conductance G which is proportional to the capacitance C of the condensor. If an alternating voltage $V_0\; e^{j\omega t}$ is applied to the inner electrode we get the following expressions for the potential, the corresponding charge and current:

$$\Phi(r,t) \;=\; \frac{V_0 ab}{b-a}\left[\frac{1}{r} - \frac{1}{b}\right] e^{j\omega t}. \qquad Q \;=\; 4\pi a^2\;\eta \;=\; 4\pi a^2\varepsilon_0\epsilon\; E_r \;=\; \varepsilon_0\Big(\varepsilon \;+\; \frac{\sigma}{j\omega\varepsilon_0}\Big)\,\frac{4\pi ab}{b-a}\; V_0\; e^{j\omega t}; \tag{14}$$

$$I \;=\; \frac{dQ}{dt} \;=\; \Big(j\omega\;\varepsilon_0\varepsilon \;+\; \sigma\Big)\,\frac{4\pi ab}{b-a}\; V_0\; e^{j\omega t} \;=\; Y\; V_0\; e^{j\omega t}, \quad \text{with} \quad Y = G + j\omega C. \tag{15}$$

So the equivalent circuit for the stationary state consists of a capacitor and a resistor in parallel. The expressions for capacitance C and the conductance G of these elements have been given in eq.(13).

3.2 More General Two-Electrode Configurations

Similar considerations apply to cases with two electrodes in less symmetric configurations as for example two spheres of different radii a and b, whose centres have a distance d. The voltages applied to these electrodes are related to the currents flowing through this electrodes by :

$$I_a \;=\; \frac{dQ_a}{dt} \;=\; Y_{aa}\; V_a\; e^{j\omega t} \;+\; Y_{ab}\; V_b\; e^{j\omega t}, \qquad I_b \;=\; \frac{dQ_b}{dt} \;=\; Y_{ba}\; V_a\; e^{j\omega t} \;+\; Y_{bb}\; V_b\; e^{j\omega t}; \tag{16}$$

with

$$Y_{ik} \;=\; G_{ik} \;+\; j\omega\; C_{ik} \quad \text{and} \quad G_{ik} \;=\; \frac{\sigma}{\varepsilon_0\varepsilon}\; C_{ik}\;. \tag{17}$$

So one knows the conductance coefficients as soon as the capacitance coefficients are given. For the two spheres mentio- ned at the beginning of this paragraph the capacitance coefficiens are computed in Noether (1961) by an infinite sequence of reflections (α_1, α_2 are the two roots of the equation $\alpha^2 \;+\; \alpha\;(a^2 + b^2 - d^2)/(ab) \;+\; 1 \;=\; 0$):

$$C_{aa}/4\pi\varepsilon_0\varepsilon \;=\; a \;+\; ab\;\sum_{k=2}^{\infty}\frac{\alpha_1 - \alpha_2}{a\;(\alpha_1^{k-1} - \alpha_2^{k-1}) \;+\; b\;(\alpha_1^{k} - \alpha_2^{k})}\;, \quad C_{ab}/4\pi\varepsilon_0\varepsilon \;=\; -\;\frac{ab}{c}\;\sum_{k=1}^{\infty}\frac{\alpha_1 - \alpha_2}{\alpha_1^{k} - \alpha_2^{k}}\;.$$

4 Two-Layer Problems

When the properties of the matter depend on position, eq.(6) must be rewritten as:

$$\nabla\cdot\Big(\sigma(\vec{r})\nabla\Phi \;+\; \varepsilon_0\varepsilon_r(\vec{r})\;\frac{\partial}{\partial t}\;\nabla\Phi\Big) \;=\; -\;\frac{\partial\rho_e}{\partial t}\;. \tag{18}$$

A frequently used special case is that where there are layers in which the matter properties are constant. Then the solutions pertaining to these different layers are described by different functions fulfilling the corresponding "Poisson equations". Their second derivatives do not exist at the interface(s). There the differential equation is replaced with two continuity conditions following from eq.(5) and from the above condition. We consider just two layers, two halfspaces, whose interface is the plane $z = 0$: The system of "Poisson equations", to which the Laplace transform has been applied, is

$$\nabla^2 \bar{\Phi}_i = -\frac{1}{\varepsilon_0 \epsilon_i}\, \bar{\rho}_e(\vec{r}, s), \quad \text{with} \quad \epsilon_i := \varepsilon_i + \sigma_i/(s\varepsilon_0). \qquad z < 0 : \varepsilon_1,\ \sigma_1; \quad z > 0 : \varepsilon_2,\ \sigma_2\ . \tag{19}$$

The two continuity conditions at the interface are (n denotes the normal to the interface) :

$$z = 0 : \Phi_1 = \Phi_2 \quad \text{and} \quad \sigma_1 \frac{\partial \Phi_1}{\partial n} + \varepsilon_0 \varepsilon_1 \frac{\partial^2 \Phi_1}{\partial t \partial n} \ = \ \sigma_2 \frac{\partial \Phi_2}{\partial n} + \varepsilon_0 \varepsilon_2 \frac{\partial^2 \Phi_2}{\partial t \partial n} \quad \text{or} \quad \epsilon_1 \frac{\partial \bar{\Phi}_1}{\partial n} = \epsilon_2 \frac{\partial \bar{\Phi}_2}{\partial n}\ . \tag{20}$$

4.1 Selfadjoint Problem and the Scalar Product

In the discussion of the selfadjointness of the two-layer problem the two continuity conditions above must be taken into consideration together with the "Poisson equations" (19) for the two media. It will be shown that the solutions can be found best by an appropriate definition of the inner product. This is not the common definition

$$(\Phi, \Psi) := (\Phi_1, \Psi_1) + (\Phi_2, \Psi_2); \quad (\Phi_1, \Psi_1) := \iint dS \int_{-\infty}^{0} dz\ \Phi_1 \Psi_1, \quad (\Phi_2, \Psi_2) := \iint dS \int_{0}^{\infty} dz\ \Phi_2 \Psi_2\ , \tag{21}$$

where dS denotes the transverse surface element (say $dS = \rho\ d\rho d\varphi$ in polar coordinates), which leads to the following form of Green's theorem:

$$(\Phi, \nabla^2 \Psi) - (\Psi, \nabla^2 \Phi) \ = \iint dS \left[\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n}\right]_{z=0} = \iint dS \left[\left(\Phi_2 \frac{\partial \Psi_2}{\partial n} - \Psi_2 \frac{\partial \Phi_2}{\partial n}\right) - \left(\Phi_1 \frac{\partial \Psi_1}{\partial n} - \Psi_1 \frac{\partial \Phi_1}{\partial n}\right)\right]_{z=0}.$$

The domain of the volume integration consists of two hemispheres (of infinite radius) touching each other along the interface $z = 0$. The surface integrals are over the two planes at $z = 0+$ and $z = 0-$ respectively, which complete these hemispheres. It is assumed that the fields vanish at infinity such that the surface integrals over the remaining surfaces are zero. The surface integral on the rhs of the above equation does not become zero, even if the continuity conditions (20) are inserted so that the operator ∇^2 is not selfadjoint. However with the following definition of the inner product

$$< \Phi, \Psi > := \epsilon_1 \iint dS \int_{-\infty}^{0} dz\ \Phi_1 \Psi_1 + \epsilon_2 \iint dS \int_{0}^{\infty} dz\ \Phi_2 \Psi_2 \ = \ \epsilon_1\ (\Phi_1, \Psi_1) + \epsilon_2\ (\Phi_2, \Psi_2) \tag{22}$$

we get

$$< \Phi, \nabla^2 \Psi > - < \Psi, \nabla^2 \Phi > = \iint dS \left[\epsilon_2 \left(\Phi_2 \frac{\partial \Psi_2}{\partial n} - \Psi_2 \frac{\partial \Phi_2}{\partial n}\right) - \epsilon_1 \left(\Phi_1 \frac{\partial \Psi_1}{\partial n} - \Psi_1 \frac{\partial \Phi_1}{\partial n}\right)\right]_{z=0} \ = \ 0. \tag{23}$$

The integrand of the surface integral becomes zero if the continuity conditions (20) are inserted; so the operator ∇^2 is selfadjoint.

4.2 Green's functions

In this paper only the Green's function for a point source in a static two-layer problem is needed. This can be found by the method of images (Jackson, 1975). Its expressions are solutions of the following equations:

$$\nabla^2 \bar{G}_{ij} \ = \ -\frac{1}{\varepsilon_0 \epsilon_i}\, \delta_{ij}\, \frac{\delta(r)}{2\pi r}\, \delta(z - z'); \qquad z = 0: \quad \bar{G}_{1j} \ = \ \bar{G}_{2j}, \quad \epsilon_1 \frac{\partial \bar{G}_{1j}}{\partial n} = \epsilon_2 \frac{\partial \bar{G}_{2j}}{\partial n}\ . \tag{24}$$

$$\bar{G}_{11} \ = \ \frac{1}{4\pi\varepsilon_0 \epsilon_1}\left(\frac{1}{R_1} - \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\frac{1}{R_2}\right), \ \bar{G}_{22} \ = \ \frac{1}{4\pi\varepsilon_0 \epsilon_2}\left(\frac{1}{R_2} - \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\frac{1}{R_1}\right), \ \bar{G}_{ij} \ = \ \frac{1}{4\pi\varepsilon_0}\frac{2}{\epsilon_1 + \epsilon_2}\frac{1}{R_j} \ \ (i \neq j). \tag{25}$$

with $R_{1,2} \ = \ \sqrt{r^2 + (z \mp z')^2}$. The functions $\bar{G}_{ij}(r, r'; z, z')$ are scalars and just different expressions representing one and the same Green's function; the first (second) subscript denotes the half-space containing the point of observation (the source point). Substituting the Green's function G for Ψ in eq.(23), eliminating the Laplacians with the help of eqs.(19) and (24) and using the definitions (21) the Laplace transform of the potential is given as:

$$\bar{\Phi}_i(\vec{r}) \ = \ \sum_{j=1}^{2}\ \left(\bar{G}_{ij}(\vec{r}, \vec{r}'), \bar{\rho}_j(\vec{r}')\right). \tag{26}$$

4.3 Fixed Point Charge Generated in an Instant

Some problems are solved, in which charges are generated or moving in one half-space, which is assumed to be an isolator, $\sigma_1 = 0$. So we have: $\epsilon_1 = \varepsilon_1; \epsilon_2 = \varepsilon_2 + \sigma/(s\varepsilon_0)$. It will turn out that the relaxation time is now given by

$$\tau_R = \varepsilon_0(\varepsilon_1 + \varepsilon_2)/\sigma. \tag{27}$$

A single point charge is generated in the isolator at the point $(r = 0, z = z' < 0)$. Though the creation of a single charge violates the law of charge conservation, such an approach is admissible in a quasi-static theory. The charge density and its transform are:

$$\rho(\vec{r}, t) = \frac{Q}{2\pi r}\,\delta(r)\,\delta(z - z'),\ \ t > 0+\,; \qquad \bar{\rho}(\vec{r}, s) = \frac{Q}{2\pi rs}\,\delta(r)\,\delta(z - z'). \quad z' < 0. \tag{28}$$

Inserting the transformed charge density into eq.(26) and doing all the evaluations including the inverse Laplace transform gives for the potential:

$$\Phi_1(r, z, t) = \frac{Q}{4\pi\varepsilon_0\varepsilon_1}\Big[\frac{1}{R_1} + \Big(\frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\,e^{-t/\tau_R} - 1\Big)\frac{1}{R_2}\Big], \qquad z \le 0; \tag{29}$$

$$\Phi_2(r, z, t) = \frac{Q}{4\pi\varepsilon_0\varepsilon_1}\,\frac{2}{\varepsilon_1 + \varepsilon_2}\,\frac{1}{R_1}\,e^{-t/\tau_R}, \qquad z \ge 0. \tag{30}$$

As in the static case also in the quasi-static case the solution is completely described by the fields (or potentials) due to the source and some image charges. In the quasi-static case all these charges are time-dependent. The fields propagate with infinite speed. Immediately after the charge has been created, a field distribution (and corresponding image charges) are established which correspond to those for two isolating half-spaces with relative dielectric constants $\varepsilon_1,\ \varepsilon_2$ respectively, which also result from the static theory (Jackson, 1975). Thereafter the conductivity starts its action to transform the field into one belonging to an ideally conducting half-space 2; the images change accordingly with the time constant τ_R.

4.4 A Point Charge Changing in Time

Now the charge fixed at $(r = 0, z = z' < 0)$ is assumed to grow from zero to the value Q during a time τ_0 and to stay constant afterwards. The charge density and its Laplace transform are:

$$\rho(\vec{r}, t) = \frac{Q}{2\pi r}\,\delta(r)\,\delta(z - z')\,\Big[\frac{t}{\tau_0}\Theta(\tau_0 - t) + \Theta(t - \tau_0)\Big],\ \ t \ge 0\,; \quad \bar{\rho}(\vec{r}, s) = \frac{Q}{2\pi rs}\,\delta(r)\,\delta(z - z')\frac{1 - e^{-s\tau_0}}{s^2\tau_0}. \tag{31}$$

The potential is computed in a manner analogous to eqs.(29) and (30):

$$\Phi_1(r, z, t) = \frac{Q}{4\pi\varepsilon_0\varepsilon_1}\Big[\Big(\frac{1}{R_1} - \frac{1}{R_2}\Big)\Big[\frac{t}{\tau_0}\,\Theta(\tau_0 - t) + \Theta(t - \tau_0)\Big] - \frac{1}{R_2}\,\frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\,\frac{\tau_R}{\tau_0}\,\big[\big(1 - e^{-t/\tau_R}\big) - \big(1 - e^{-(t-\tau_0)/\tau_R}\big)\Theta(t - \tau_0)\,\big]\Big], \qquad z \le 0; \tag{32}$$

$$\Phi_2(r, z, t) = \frac{Q}{4\pi\varepsilon_0\varepsilon_1}\,\frac{1}{R_1}\,\frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\,\frac{\tau_R}{\tau_0}\,\Big[\big(1 - e^{-t/\tau_R}\big) - \big(1 - e^{-(t-\tau_0)/\tau_R}\big)\,\Theta(t - \tau_R)\Big], \qquad z \ge 0. \tag{33}$$

The potential now depends on two rates, the growth rate τ_0 and the decay rate τ_R; their ratio has a strong influence on the shape of the time dependence of the field. If $\tau_R \ll \tau_0$ then the image charge located at $z = -z' > 0$ giving the field in halfspace 2 behaves the same way as the primary charge as long as the latter increases; when the latter stays constant, the former slowly decreases reflecting the slow decay of the field within the weakly conducting dielectric. The lossy dielectric behaves like a circuit, which gives the derivative of the primary field. In fact:

$$\Phi_2(r, z, t) \propto (1 - e^{-t/\tau_R}) \approx 1 \quad \text{for} \quad \tau_R \ll t \le \tau_0; \quad \Phi_2(r, z, t) \propto \big(e^{-t/\tau_R} - e^{-(t-\tau_0)/\tau_r}\big) \approx 0 \quad \text{for} \quad \tau_R \ll \tau_0 \ll t.$$

4.5 A Moving Point Charge

A charge Q starts at time $t = 0$ at the point $z = z_0 < 0$ and moves with constant speed v_0 down the negative z-axis. The charge density and its transform are:

$$\rho(\vec{r}, t) = \frac{Q}{2\pi r}\,\delta(r)\,\delta(z - z_0 + v_0 t), \qquad \bar{\rho}(\vec{r}, s) = \frac{Q}{2\pi r v_0}\,\delta(r)\,\Theta(z_0 - z)\,e^{-s(z-z_0)/v_0}. \tag{34}$$

The potential is computed by the same methods as above:

$$\Phi_1(r,z,t) = \frac{Q}{4\pi\varepsilon_0\varepsilon_1}\left[\frac{1}{R_1(t)} - \frac{1}{R_2(t)}\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2} - \frac{2}{(\varepsilon_1+\varepsilon_2)\,\tau_R}\int_0^t dt'\,\frac{1}{R_2(t')}\,e^{-(t-t')/\tau_R}\right], \quad z \leq 0; \tag{35}$$

$$\Phi_2(r,z,t) = \frac{Q}{4\pi\varepsilon_0\varepsilon_1}\,\frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\left[\frac{1}{R_1(t)} - \frac{1}{\tau_R}\int_0^t dt'\,\frac{1}{R_1(t')}\,e^{-(t-t')/\tau_R}\right], \quad z \geq 0 \tag{36}$$

with $R_{1,2}(t) := \sqrt{r^2 + (z \mp z_0 \pm v_0 t)^2}$. The moving primary charge induces image charges at the corresponding image points similar to the two cases treated above; that in the uper halfspace (2) moves in the opposite direction. In addition there is also an integral over all previous times, which represents a trail of the field shed by the motion of the image charge through the lossy dielectric.

4.6 A Model for Resistive Plate Chambers, Comparison of the Two Theories

An important application of the theory described above is to Resistive Plate Chambers (RPC's), Crotty et al. (1995), Heubrandtner et al. (1998). This model is an infinite plane condensor. The space between the electrodes is filled by two layers; one is vacuum representing the gas gap of the RPC, in which the electron cloud is generated, which induces the signals on the anode strip. The other layer is a weakly conducting dielectric representing a pane of melamine-phenolic laminate or glass, which confines the gas gap and supports the anode strips. The problem has been treated by an approximate dynamic theory starting from the full Maxwell equations and the resulting series solutions for the signal strength have been summed numerically (Schöpf and Schnizer, 1992). Now the problem has been treated again by the quasi-static theory in a much simpler way (Heubrandtner, 1999). The advantages are serveral: The expressions are simpler, the set of continuity conditions at the interface between the two layers contains 2 in place of 4 conditions, the resulting series solutions are simpler and amenable to convergence acceleration (Weniger (1989), Singh et al. (1990), Krebs (1997)). Heubrandter (1999) compares the signals computed by the dynamics and the quasi-static theory. The curves are indistinguishable for the conductivity range $0 \leq \sigma \leq 10^{-3}\ S/m$. The analogous model describing a counter, namely a coaxial circular tube with an inner lining consisting of a weakly conducting dielectric has also been solved; results will be published elsewhere.

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Schrifttum

Crotty, I., Cerron Zeballos, E., Lamas Valverde, J., Hatzifotiadou, D., Williams, M. C. S. and Zichichi, A., The Wide Gap Resistive Plate Chamber, Nucl. Instr. and Meth., A 360, 512 - 520, 1995.

Dědek, L., Bachorec, T., Evaluation of transient electric field in lossy media by means of ANSYS dynamical module. Proc. 8th Int. IGTE Symp. on Num. Field Calc. in Elect. Eng., September 21-24, 1998, Graz. 322-326.

Dědek, L.: Signals Induced by Single Charge in the Resistive Plate Chamber. Proc. Measurement99 Conf., 1999, Smolenice, Slovakia, p. 285 - 288.

Doetsch, G., Anleitung zum praktischen Gebrauch der Laplace-Transformation und der Z-Transformation, 6. Aufl., Oldenbourg Verlag, Muenchen, Wien, 1989.

Fano, R.M., Chu, L.J., Adler, R.B., Electromagentic fields, energy, and forces. Wiley & Sons, New York, 1963.

Haus, H.A., Melcher, J.R., Elelctromagentic fields and energy. Prentice Hall Inc., Englewood Cliffs, NJ, 1989.

Heubrandtner,Th., Schnizer, B., Schöpf, H., Signals in a Resistive Plate Chamber. Kleinheubacher Berichte 41, 484 - 489, 1998a .

Heubrandtner,Th., Schnizer, B., Schöpf, H., A simple theory for signals induced by a point charge moving in a resistive plate chamber. Nucl.Inst.Meth. Phys. Res. A419, 721-725, 1998b .

Heubrandtner, Th., Theoretical Models for Signal Generation in Resistive Plate Chambers. Dissertation, Technisch-Naturwissenschaftliche Fakultät, Technische Universität Graz, 1999.

Jackson, J.D., Classical Electrodynamics. Wiley, New York, 1975, Sect.4.4.

Krebs, V., Sphärische Multipolentwicklung von Beugungsfeldern des elliptischen Kegels und ihre numerische Auswertung mit Hilfe von Reihentransformationen. Dissertation, Fakultät für Elektrotechnik, Ruhr-Universität Bochum, 1997.

Noether, F., Das stationäre (und quasistationäre) elektromagnetische Feld. in: Ph. Frank, F. Mises (Ed.) Die Differential und Integralgleichungen der Mechanik und Physik. 2. Aufl. Vieweg, Braunschweig, 1961, pp. 642-655.

Schöpf, H., and Schnizer, B., Signals generated by a uniformly moving charge in a counter. Kleinheubacher Berichte, 35, 645-655, 1992 a.
Note: An serious error is in the equation preceeding eq.(27). This is corrected in 1992 b.

Schöpf, H., and Schnizer, B., Theory Describing Cathode Signals from Charges Moving in Counters with a Poorly Conducting Cathode, Nucl. Instr. and Meth. A 323, 338 - 344, 1992 b.

Singh, S., Richards, W.F., Zinecker, J.R., Wilton, D.R., Accelerating the convergence of series representing the free space periodic Green's function. IEEE Trans. AP38, 1958-1962, 1990.

Stratton, J.A., Electromagnetic Theory. McGraw-Hill, New York, 1941. p.15.

Weniger, J., Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series. Computer Physics Reports 10 (1989) 189 - 371.

Wolfram, St., The Mathematica Book, Cambridge Univ.Press, Cambridge 1996.