14. Vector Algebra and Vector Analysis

2016-07-07

$Version

```
10.0 for Mac OS X x86 (64-bit) (December 4, 2014)
```

Mathematica 9: The commands for vector analysis are contained in the kernel in contrast to earlier versions.

Mathematica 7 or 8: The package must be loaded by the command `<< "VectorAnalysis`"` . Some of the commands differ from those given in version 9, so in these lecture notes. In particular, the arguments of the operators do not contain the list of the independent variables as arguments.

Ph. Boylard: A Guide to Standard Mathematica Packages. *Mathematica* Technical Report (1991).
E. Martin: *Mathematica 3.0* Standard Add-On Packages. Cambridge University Press. pp.74-82.

14.1 Rectangular, Cylindrical and Spherical Coordinates

Rectangular (as well as cylindrical and spherical) coordinates have been treated at the end of Chap.3.
Knowledge of these subjects is pressuposed in this chapter.

14.2   Curvilinear Orthogonal Coordinates

14.2.1 Literature

Murray R. Spiegel, Schaum's Mathemtical Handbook of Formulas and Tables,
     McGraw-Hill, New York 1968, pp.116-130
Philip M. Morse, Herman Feshbach: Methods of Theoretical Physics,
     McGraw-Hill, New York, 1953. (MF), 656ff.
George B. Arken, Mathematical Methods for Pysicists, 3 rd ed., Academic Press, New York, 1985.
B. Schnizer, Analytical methods in applied theoretrical physics. Lecture notes (in German). Chaps. 6 and 7.
     http://itp.tugraz.at/~schnizer/AnalyticalMethods/
P. Moon, D.E. Spencer, Field Theory Handbook. 2 nd ed., Springer, 1988. (MS)

G. Wendt, Statische Felder und Stationäre Ströme. Hb. d. Physik XVI Springer, 1958, S.1 ff.
W.R. Smith, Static and Dynamic Electricity, McGraw-Hill, 1950

14.2.2 Covariant and contravariant coordinates

A general non-singular transformation from Cartesian coordinates x_1, x_2, x_3 to more general curvilinear coordinates u_1, u_2, u_3 is given by:

$$x_i = x_i(u_1, u_2, u_3), \qquad u_k = u_k(x_1, x_2, x_3); \qquad i, k = 1, 2, 3. \quad (1)$$

So a triple of numbers for the u_k's gives a point in space in the same way as a triple of Cartesian coordinates x_1, x_2, x_3. Keeping two u_k's constant while varying the third one gives a curve in space.
There are three such curves passing through each point. The three tangents to these curves are given by:

$$\overset{k}{t_i} = \partial x_i / \partial u_k; \qquad\qquad (2)$$

k numbering the curves; i numbering the components of each vector. These three vectors are orthogonal to each other, but they are not normalized; in addition, their directions and lengths may change from point to point.

Varying two u_k's for a fixed third u_s gives a surface in space. There are three surfaces passing through a given point with coordinates u_1, u_2, u_3. The three gradients

$$\overset{k}{\gamma}_i = \partial u_k / \partial x_i; \tag{3}$$

(k numbering the surface; i numbering the components of each vector) make up a second triple of orthogonal non-normalized vectors. The two triples are reciprocal to each other, i.e.

$$\overset{p}{t}_i \overset{q}{\gamma}_i = \delta_{pq}. \tag{4}$$

The inner products of the tangent vectors give the covariant metric tensor

$$g_{pq} := \overset{p}{t}_i \overset{q}{t}_i = (\partial x_i / \partial u_p)(\partial x_i / \partial u_q) = g_{qp}. \tag{5}$$

The inner products of the gradient vectors give the contravariant metric tensor

$$g^{pq} := \overset{p}{\gamma}_i \overset{q}{\gamma}_i = (\partial u_p / \partial x_i)(\partial u_q / \partial x_i) = g^{qp}. \tag{6}$$

So in non-orthogonal coordinate systems there are two systems, whose base vectors may change their orientations and lengths from point to point. The components of any vector **a**, whose Cartesian components are a_i:

$$\mathbf{a} = a_i e_i$$

may be taken with respect to each of these systems. The covariant components A_p are:

$$a_i = A_p \overset{p}{\gamma}_i = A_p \partial u_p / \partial x_i, \qquad A_p := A_i \overset{k}{t}_i = A_i \partial x_i / \partial u_k; \tag{7}$$

while the contravariant components A^p are:

$$a_i = A^p \overset{p}{t}_i = A^p \partial x_i / \partial u_k, \qquad A^p := a_i \overset{k}{\gamma}_i = a_i \partial u_k / \partial x_i. \tag{8}$$

The arc element may be expressed by the covariant metric tensor (5):

$$ds^2 := dx_i dx_i = g_{pq} \, du_p \, du_q. \tag{9}$$

14.2.3 Introducing physical coordinates for orthogonal curvilinear systems

In orthogonal curvilinear systems, the base vectors $\overset{p}{\gamma}_i$ and $\overset{p}{t}_i$ have the same directions. One introduces one set of normalized base vectors e_i:

$$e_i := \overset{p}{t}_i / h_p = \overset{p}{\gamma}_i h_p$$

In place of the covariant and contravariant components of a vector one gets one set, the physical coordinates q_i such that :

$$\mathbf{a} = a_i e_i = q_i e_i = A^p \overset{p}{t} = A_p \overset{p}{\gamma}; \tag{10}$$

$$q_i = h_i A^i = A_i / h_i. \qquad \text{(No summation in this line !)}$$

For example, in cylindrical coordinates ρ, ϕ, z the transformation from rectangular coordinates, the three tangents to the coordinate curves and the gradients of the coordinate surfaces are:

$$x = x_1 = \rho \cos\phi, \ \ y = x_2 = \rho \sin\phi, z = x_3; \ \ \rho = \sqrt{x^2 + y^2}, \ \phi = \tan\frac{y}{x}, \ z = x_3.$$

$$\overset{k}{t_i} = \ \partial x_i / \ \partial u_k: \ \ \overset{1}{t_i} = (\ \cos\phi, \ \sin\phi, \ 0), \ \ \overset{2}{t_i} = (\ -\rho \ \sin\phi, \ \rho \cos\phi, \ 0), \ \ \overset{3}{t_i} = (0, \ 0, \ 1), \text{ with } \rho = \sqrt{x^2 + y^2},$$

$$\overset{k}{\gamma_i} = \ \partial u_k / \ \partial x_i; \ \ \overset{1}{\gamma_i} = (x/\rho, \ y/\rho, \ 0), \overset{2}{\gamma_i} = (-y, \ x)/(x^2 + y^2), \ \ \ \ \overset{3}{\gamma_i} = (0, \ 0, \ 1).$$

The metric coefficients h_i are:
$$h_\rho = h_z = 1, \ h_\phi = \rho.$$

The various coordinates of the position (**r**) and of the velocity vector (**v**) are:

$\text{r} = \ r_\rho e\rho + \ r_\phi e\phi + \ r_z z ez, \ \ \ \text{r}_\rho = \rho, \ \ r_\phi = 0, \ \ \ r_z = z. \ A^\rho = A_\rho = 0, \ A^\phi = A_\phi = 0, \ A^z = A_z = z.$

v $= \ v_\rho e\rho + \ v_\phi e\phi + \ v_z z ez, \ \ \text{v}_\rho = \dot\rho, \ \text{v}_\phi = \rho\dot\phi, \ \text{v}_z = \dot z. \ A^\rho = A_\rho = \dot\rho, A^\phi = \dot\phi, A_\phi = \rho\dot\phi, A^z = A_z = z.$

The square of the arc element is:

$$ds^2 = dx^2 + dy^2 + dz^2 = \ h_1^2 \, du_1^2 + h_2^2 \, du_2^2 + \ h_3^2 \, du_3^2. \ \ \ \ \ \ \ \ (11)$$

The surface elements and the volume element are:
$$df_1 = h_2 h_3 du_2 du_3, \ \ df_2 = h_1 h_3 \ du_1 du_3, \ \ df_3 = h_1 h_2 \ du_1 du_2; \ \ \ \ \ \ \ (12)$$

$$dV = \ \ h_1 \ h_2 \ h_3 \ du_1 \ du_2 \ du_3. \ \ \ \ \ \ \ \ (13)$$

14.2.4 Vector Analysis implementations in *Mathematica*

The vector analysis impemented in Mathematica deals only with orthogonal coordinate systems, so only with physical coordinates as explained in the previous subsection.

As of *Mathematica* 9 there are available two versions of implementations :
  1. The old version as implemented in version 8 and previous ones with some modifications is available as a package, which must be called by the command:

Needs["VectorAnalysis`"]

This implementaton is created for 14 curvilinear orthogonal coordinate systems in three-dimensional
Euclidean space given below under 14.3, in particular 14.3.5.

When the package is called then the new vector analysis functionality implemented in the kernel (see 2.)
is blocked.

  2. *The new version of Mathematica* Vector Analysis is now implemented in the kernel; so it is ready
automatically if not blocked (see previous paragraph 1.) It used new commands completely different
from those employed in the package. It has implementations for two-dimensional curvilinear orthogonal
coordinate systems. It treats the three-dimensional systems listed in 1.) In addition it deals with some
n-dimensional orthogonal systems. Besides these cases it deals also with vector analysis in non-
Euclidean space, as e.g. geometry on the surface of a sphere. More details will be given below
under

B),

A) VectorAnalysis using the package

$Version

10.0 for Mac OS X x86 (64-bit) (September 10, 2014)

The package is called by

<< VectorAnalysis`

or

Needs["VectorAnalysis`"];

14.3   Coordinate systems implemented in the package VectorAnalysis

The coordinate systems implemented in the package are the following 14 systems:

Bipolar	Bispherical
Cartesian	ConfocalEllipsoidal
ConfocalParaboloidal	Conical
Cylindrical	EllipticCylindrical
OblateSpheroidal	ParabolicCylindrical
Paraboloidal	ProlateSpheroidal
Spherical	Toroidal

These embrace the 11 systems in which the 3D Helmholtz equation $(\Delta + k^2)\ \Psi = 0$ can be solved by separation of variables $\Psi(u_1, u_2, u_3) = U_1(u_1)\ U_2(u_2)\ U_3(u_3)$; i.e. the partial differential equation can be decomposed into a system of 3 common differential equations, each one for exactly one dependent variable U_i depending on only one independent variable u_i . Such an "ansatz" is also usable to separate the corresponding potential equation $\Delta\Psi = 0$. In addition, there is a number of potential equations, which are separable by R-separation: The potential equation $\Delta\Psi = 0$ is transformed into a new linear partial differential equation L($q(u_1, u_2)\ \Psi(u_1, u_2, u_3)$)) by introducing a new dependent variable.The factor $q(u_1, u_2)$ is chosen conveniently such that the new differential operator L is eparabel. The three systems **Bipolar, Bispherical** and **Toroidal** are such systems, while in the 11 othersystems listed above the Helmholtz and the potential equations are separable.

When one startes the package then the Cartesian system is the default system.In this version of *Mathematica* one can select a default system. As long as one uses only operations in that system one needs not give instructions concerning this system in the commands. One can use also commands employing variables of an other systema (without leaving the selected default system); but then this command must also contain the name of this extraneous system and also the names of the variables.

14.3.1 Selecting a default coordinate system

> SetCoordinates[*coordsys*[c_1, c_2, c_3]]
> sets the default coordinate system to be *coordsys* with variables c_1, c_2, and c_3.

For example, circular cylindrical coordinates r,φ,z are selected in the following way:

<< VectorAnalysis`

```
SetCoordinates[Cylindrical[r,φ,z]]
```

Cylindrical[r, φ, z]

14.3.2 Transformations between Systems

$$x_i = x_i(\textbf{\textit{pt}}), \qquad \textbf{\textit{pt}} = (u_1, u_2, u_3) = (u_1, u_2, u_3)[\textbf{x, y, z}]$$

CoordinatesToCartesian[*pt*]	give the Cartesian coordinates of point ***pt***, which is given in default coordinates
CoordinatesToCartesian[*pt*, *coordsys*] which	give the Cartesian coordinates of point ***pt***, is given in the coordinate system ***coordsys***
CoordinatesFromCartesian[*pt*]	give the coordinates in the default coordinate system of point ***pt***, whose Cartesian coordinates are given
CoordinatesFromCartesian[*pt*, *coordsys*]	give the coordinates in the coordinate system coordsys of point ***pt***, which is given in Cartesian coordinates

```
CoordinatesToCartesian[{r,φ,z}]
```

{r Cos[φ], r Sin[φ], z}

```
CoordinatesToCartesian[{5., N[Pi] /3., 2.}]
```

{2.5, 4.33013, 2.}

```
CoordinatesFromCartesian[{x,y,z}]
```

$\left\{ \sqrt{x^2 + y^2}, \text{ArcTan}[x, y], z \right\}$

Note: ArcTan[x,y] works correctly in all four quadrants of the x, y-plane; whereas ArcTan[y/x] is limited
to the 1st and 4th quadrant (principal branch of ArcTan[xx]).

```
CoordinatesFromCartesian[{1.,1.,2}]
```

```
N[Pi]/4.
```

14.3.3 Vector Algebra

```
Needs["VectorAnalysis`"];

a = {ar, aφ, az}; b = {br, bφ, bz}; c = {cr, cφ, cz};

SetCoordinates[Cylindrical[r,φ,z]]
```

Cylindrical[r, φ, z]

14.3.3.1   Scalar product of two vectors

```
a.b                                    (* correct *)
```

$ar\,br + az\,bz + a\varphi\,b\varphi$

```
DotProduct[a,b]                        (*  wrong  *)
```

$az\,bz + ar\,br\,Cos[a\varphi]\,Cos[b\varphi] + ar\,br\,Sin[a\varphi]\,Sin[b\varphi]$

```
DotProduct[a,b,Cartesian]              (* correct *)
```

$ar\,br + az\,bz + a\varphi\,b\varphi$

```
DotProduct[a,b,Cylindrical]            (*  wrong  *)
```

$az\,bz + ar\,br\,Cos[a\varphi]\,Cos[b\varphi] + ar\,br\,Sin[a\varphi]\,Sin[b\varphi]$

```
DotProduct[a,b,Spherical]              (*  wrong  *)
```

$ar\,br\,Cos[a\varphi]\,Cos[b\varphi] + ar\,br\,Cos[az]\,Cos[bz]\,Sin[a\varphi]\,Sin[b\varphi] +$
$\;ar\,br\,Sin[az]\,Sin[a\varphi]\,Sin[bz]\,Sin[b\varphi]$

14.3.3.2   Vector product

```
Cross[a, b]                            (* correct *)
```

$\{a\varphi\,bz - az\,b\varphi,\; az\,br - ar\,bz,\; -a\varphi\,br + ar\,b\varphi\}$

```
CrossProduct[a,b,Cartesian]            (* correct *)
```

$\{a\varphi\,bz - az\,b\varphi,\; az\,br - ar\,bz,\; -a\varphi\,br + ar\,b\varphi\}$

```
CrossProduct[a,b]                      (*  wrong  *)
```

$\left\{\sqrt{\left((-ar\,bz\,Cos[a\varphi] + az\,br\,Cos[b\varphi])^2 + (ar\,bz\,Sin[a\varphi] - az\,br\,Sin[b\varphi])^2\right)},\right.$
$\;ArcTan[ar\,bz\,Sin[a\varphi] - az\,br\,Sin[b\varphi],\, -ar\,bz\,Cos[a\varphi] + az\,br\,Cos[b\varphi]],$
$\left.\;-ar\,br\,Cos[b\varphi]\,Sin[a\varphi] + ar\,br\,Cos[a\varphi]\,Sin[b\varphi]\right\}$

```
CrossProduct[a,b,Cylindrical]          (*  wrong  *)
```

$\left\{\sqrt{\left((-ar\,bz\,Cos[a\varphi] + az\,br\,Cos[b\varphi])^2 + (ar\,bz\,Sin[a\varphi] - az\,br\,Sin[b\varphi])^2\right)},\right.$
$\;ArcTan[ar\,bz\,Sin[a\varphi] - az\,br\,Sin[b\varphi],\, -ar\,bz\,Cos[a\varphi] + az\,br\,Cos[b\varphi]],$
$\left.\;-ar\,br\,Cos[b\varphi]\,Sin[a\varphi] + ar\,br\,Cos[a\varphi]\,Sin[b\varphi]\right\}$

14.3.3.3   Scalar triple product (German: Spatprodukt)

```
a.Cross[b, c]                          (* correct *)
```

$a\varphi\,(bz\,cr - br\,cz) + az\,(-b\varphi\,cr + br\,c\varphi) + ar\,(b\varphi\,cz - bz\,c\varphi)$

ScalarTripleProduct[a,b,c, Cartesian] (* **correct** *)

aφ bz cr − az bφ cr − aφ br cz + ar bφ cz + az br cφ − ar bz cφ

ScalarTripleProduct[a,b,c] (* **wrong** *)

− ar br cz Cos[bφ] Sin[aφ] + ar bz cr Cos[cφ] Sin[aφ] + ar br cz Cos[aφ] Sin[bφ] −
az br cr Cos[cφ] Sin[bφ] − ar bz cr Cos[aφ] Sin[cφ] + az br cr Cos[bφ] Sin[cφ]

14.3.4 Vector Analysis

`Grad[s]`	give the gradient of the scalar field **s** in the default coordinate system
`Div[v]`	give the divergence of the vector field **v** in the default coordinate system
`Curl[v]`	give the curl (rotor) of the vector field **v** in the default coordinate system
`Laplacian[s]`	give the Laplacian of the scalar field **s** or the in the default coordinate system
`Laplacian[v]`	give the Laplacian of the vector field **v** in the default coordinate system
`Biharmonic[s]`	give the Laplacian of the Laplacian of the scalar field **s** in the default coordinate system
`Grad[s, coordsys], Div[v, coordsys], ...`	give the result for the coordinate system **coordsys**

`Needs["VectorAnalysis`"];`

General::obspkg : VectorAnalysis` is now obsolete. The legacy version being loaded may conflict with current functionality. See the Compatibility Guide for updating information.

`SetCoordinates[Cylindrical[r,φ,z]]`

Cylindrical[r, φ, z]

14.3.4.1   Gradient of a scalar field

`Grad[ψ[r,φ,z]]`

$$\left\{\psi^{(1,0,0)}[r, \varphi, z], \frac{\psi^{(0,1,0)}[r, \varphi, z]}{r}, \psi^{(0,0,1)}[r, \varphi, z]\right\}$$

`Grad[ψ[r,ϑ,φ], Spherical[r,ϑ,φ]]`

$$\left\{\psi^{(1,0,0)}[r, \vartheta, \varphi], \frac{\psi^{(0,1,0)}[r, \vartheta, \varphi]}{r}, \frac{\text{Csc}[\vartheta] \, \psi^{(0,0,1)}[r, \vartheta, \varphi]}{r}\right\}$$

14.3.4.2   Divergence of a vector field

`Clear[a,b,c];`
`a = {ar[r,φ,z], aφ[r,φ,z], az[r,φ,z] };`

`Div[a]//Expand`

$$\frac{ar[r, \varphi, z]}{r} + az^{(0,0,1)}[r, \varphi, z] + \frac{a\varphi^{(0,1,0)}[r, \varphi, z]}{r} + ar^{(1,0,0)}[r, \varphi, z]$$

14.3.4.3   Curl of a vector field

`Curl[a]//Expand`

$$\left\{-a\varphi^{(0,0,1)}[r, \varphi, z] + \frac{az^{(0,1,0)}[r, \varphi, z]}{r}, ar^{(0,0,1)}[r, \varphi, z] - az^{(1,0,0)}[r, \varphi, z],\right.$$

$$\left.\frac{a\varphi[r, \varphi, z]}{r} - \frac{ar^{(0,1,0)}[r, \varphi, z]}{r} + a\varphi^{(1,0,0)}[r, \varphi, z]\right\}$$

14.3.4.4 Laplacian of a scalar field

```
Expand[Laplacian[ ψ[r,φ,z] ]]
```

$$\psi^{(0,0,2)}[\mathbf{r}, \varphi, \mathbf{z}] + \frac{\psi^{(0,2,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} + \frac{\psi^{(1,0,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}} + \psi^{(2,0,0)}[\mathbf{r}, \varphi, \mathbf{z}]$$

14.3.4.5 Vector Laplacian = Laplacian of a vector field

$$\Delta\vec{a} = \text{grad}(\text{div}\vec{a}) - \text{rot}(\text{rot}\vec{a})$$

follows from: $\text{rot}(\text{rot}\vec{a}) = [\nabla,[\nabla\vec{a}]] = \nabla(\nabla, \vec{a}) - (\nabla,\nabla)\vec{a} = \text{grad}(\text{div}\vec{a}) - \Delta\vec{a} = \text{grad}(\text{div}\vec{a}) - \Delta\vec{a}$

```
Expand[ Grad[Div[a]] - Curl[Curl[a]]  ]
```

$$\Big\{ -\frac{\text{ar}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} + \text{ar}^{(0,0,2)}[\mathbf{r}, \varphi, \mathbf{z}] - \frac{2\,\text{a}\varphi^{(0,1,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} + \frac{\text{ar}^{(0,2,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} +$$

$$\frac{\text{ar}^{(1,0,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}} + \text{ar}^{(2,0,0)}[\mathbf{r}, \varphi, \mathbf{z}], \; -\frac{\text{a}\varphi[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} + \text{a}\varphi^{(0,0,2)}[\mathbf{r}, \varphi, \mathbf{z}] +$$

$$\frac{2\,\text{ar}^{(0,1,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} + \frac{\text{a}\varphi^{(0,2,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} + \frac{\text{a}\varphi^{(1,0,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}} + \text{a}\varphi^{(2,0,0)}[\mathbf{r}, \varphi, \mathbf{z}],$$

$$\text{az}^{(0,0,2)}[\mathbf{r}, \varphi, \mathbf{z}] + \frac{\text{az}^{(0,2,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}^2} + \frac{\text{az}^{(1,0,0)}[\mathbf{r}, \varphi, \mathbf{z}]}{\mathbf{r}} + \text{az}^{(2,0,0)}[\mathbf{r}, \varphi, \mathbf{z}] \Big\}$$

The third component of the Vector Laplacian applied to the vector **a** equals the expression
Laplacian[az] as in Cartesian coordinates (cf. § 3.14). However, in the first two components
of the vector Laplacian the two first components of the vector **a** are mixed.
The operator Laplacian[] applied to vector fields works also properly:

```
Laplacian[a] - % // Simplify
```

$\{0, 0, 0\}$

14.3.5 Curvilinear Systems Implemented in Mathematica.

The systems of orthogonal curvilinear systems implemented in the package "Calculus`VectorAnalysis`"
(now: "VectorAnalysis`") (Guide to Standard Packages, p.26f, Standard Add-on Packages, p. 75)
are listed below and in more detail in Table 14.1.
This table can be found in the File Table14-1.pdf in the directory *Mathematica* Course notes.

Bipolar	Bispherical
Cartesian	ConfocalEllipsoidal
ConfocalParaboloidal	Conical
Cylindrical	EllipticCylindrical
OblateSpheroidal	ParabolicCylindrical
Paraboloidal	ProlateSpheroidal
Spherical	Toroidal

Tab. 14.1 Curvilinear Systems Implemented in the *Mathematica* package "VectorAnalysis"

System	MFNr	MFp	MSNr	Sep.	MaxSep	*Mathematica* command
Bipolar			Fig.2.09	RS		SetCoordinates[Bipolar[u,v,z,a]]
Bispherical		665	T4.03	RS		SetCoordinates[Bispherical[θ,η,ψ, a]]
Cartesian	I	656	T1.01	HS	MC	SetCoordinates[Cartesian[x,y,z]]
ConfocalEllipsoidal	X	663	T1.10	HS		??
ConfocalParaboloidal	XI	664	T1.11	HS		??
Conical	VI	659	T1.09	HS	MR	SetCoordinates[Conical[r,θ,λ, b,c]]
Cylindrical	II	656	T1.02	HS	MC	SetCoordinates[Cylindrical[r,ψ,z]]
EllipticCylindrical	III	657	T1.03	HS	MC	SetCoordinates[EllipticCylindrical[η,ψ ,z,a]]
OblateSpheroidal	IX	662	T1.07	HS		SetCoordinates[OblateSpheroidal[$\theta + \frac{\pi}{2},\eta,\psi$,a]]
ParabolicCylindrical	IV	658	T1.04	HS	MC	SetCoordinates[ParabolicCylindrical[μ,ν ,z]]
Paraboloidal	VII	660	T1.08	HS		SetCoordinates[Paraboloidal[μ,ν,ψ]]
ProlateSpheroidal	VIII	661	T1.06	HS		SetCoordinates[ProlateSpheroidal[η,θ,ψ ,a]]
Spherical	V	658	T1.02	HS	MR	SetCoordinates[Spherical[r,θ,ψ]]
Toroidal		666	T4.03	RS		SetCoordinates[Toroidal[η,θ,ψ ,a]]

Abbreviations:

MF = Morse, Feshbach: Methods of Theoretical Physics, s. beginning of 14.2.

MFNr. in Roman numerals gives the number in the list of coordinate systems provided in MF.

MFp gives the page of MF, where this system is treated.

MS = Moon, Spencer: Field Theory Handbook, s. beginning of 14.2.

Tn.k gives the number of the corresponding table in MS. The arguments in SetCoordinates[] above have been chosen such that the formulas are obtained as given in MS.

HS = In this system the 3D Hemholtz, so also the 3D potential equation, can be solved by separation of variables.

RS = The potential equation is R-separable in this system.

MC = In this coordinate system the Maxwell curl equations can be solved by one component electric or magnetic Hertz vectors aligned with the generators of the cylindrical coordinate surfaces. The corresponding solutions are given by the vector fields **L,M,N** (s.14.3).

MR = In these two coordinate system the Maxwell curl equations can be solved by one component electric or magnetic Hertz pointing into the radial direction. The corresponding solutions are given by the vector fields **L,M,N** (s.14.3).

Important information on these systems and the corresponding coordinates are obtained by the following
commands.

`CoordinateRange[]`	give the intervalls over which the coordinates of the default coordinate system may vary
`Parameters[ ]`	give a list of the default values for the parameters of the default coordinate system
`ParameterRanges[]`	give the intervals over which each of the parameters of the default coordinate system may range
`CoordinateRange[coordsys], Paramters[coordsys],` `ParameterRanges[coordsys]`	give the result for the coordinate system ***coordsys***
`SetCoordinates[coordsys[varnames, param]]`	set the default coordinate system to be ***coordsys*** with variable names **varnames** and parameter values **param**

`Needs["VectorAnalysis`"]`

`SetCoordinates[Cylindrical[r,φ,z]]`

Cylindrical[r, φ, z]

Paraboloidal Coordinates μ, ν, ψ are defined in Moon Spencer in Table 1.08, pp. 34 f.

`SetCoordinates[Paraboloidal[μ,ν,ψ]]`

Paraboloidal[μ, ν, ψ]

One set of coordinates surfaces is given by paraboloids of rotation open upwards, while the parameter μ varies in the range $0 < \mu < \infty$, the other one by $0 < \nu < \infty$ giving similar surfaces open downwards. The azimuth varies within $-\pi \le \psi \le \pi$.

`CoordinatesToCartesian[{μ,ν,ψ}]`

$\left\{ \mu \nu \text{ Cos}[\psi], \mu \nu \text{ Sin}[\psi], \frac{1}{2} \left(\mu^2 - \nu^2 \right) \right\}$

`CoordinateRanges[]`

$\{ 0 \le \mu < \infty, 0 \le \nu < \infty, -\pi \le \psi \le \pi \}$

The paraboloidal coordinate system has no free parameters:

`ParameterRanges[]`

Prolate Spheroidal Coordinates η, θ, ψ are defined in Moon Spencer in Table 1.06, pp. 28 f.

`SetCoordinates[ProlateSpheroidal[]]`

ProlateSpheroidal[Xxi, Eeta, Pphi, 1]

`ProlateSpheroidal[η, θ, ψ]`

ProlateSpheroidal[η, θ, ψ]

One set of coordinate surfaces are prolate ellipsoids of rotation; each one corresponding to a single value Xxi (= η) taken from $0 \le$ Xxi $< \infty$; while Eeta (= θ) runs from 0 to π ($0 \le \theta \le \pi$)

and $-\pi < \psi \leq \pi$. The equal semi-axes are smaller than the third semi-axis || to the z-axis.
A single value of Eeta (= θ) $0 < \eta < \pi$ gives a hyperboloid of rotation consisting of two sheets
as it is obtained by rotating a hyperbola around an axis passing through the foci. The azimuth
varies within $-\pi < \psi \leq \pi$. A half plane bounded by the z-axis belongs to each value of ψ.
There is an infinite number of such coordinate systems. Each system corresponds to
a given value of the parameter **e (= a)** determining the distance of the foci common to all
conics of the system from the origin. The default value of **e** is unity as selected above by default.

SetCoordinates[ProlateSpheroidal[η, θ, ψ, e]]

ProlateSpheroidal[η, θ, ψ, e]

CoordinatesToCartesian[{η,θ,ψ}]

{e Cos[ψ] Sin[θ] Sinh[η], e Sin[θ] Sin[ψ] Sinh[η], e Cos[θ] Cosh[η]}

CoordinateRanges[]

{0 ≤ η < ∞, 0 ≤ θ ≤ π, $-\pi$ < ψ ≤ π}

The parameter e may be any positive real number:

ParameterRanges[]

0 < #1 < ∞

CoordinatesFromCartesian[{x, y, z}]

$$\left\{ \text{Re}\left[\text{ArcCosh}\left[\frac{\mathbb{i} \sqrt{x^2 + y^2} + z}{e} \right] \right], \text{Im}\left[\text{ArcCosh}\left[\frac{\mathbb{i} \sqrt{x^2 + y^2} + z}{e} \right] \right], \text{ArcTan}[x, y] \right\}$$

Oblate Spheroidal Coordinates ξ, η, ϕ are defined in Moon Spencer in Table 1.07, pp. 31 f.

SetCoordinates[OblateSpheroidal[]]

OblateSpheroidal[Xxi, Eeta, Pphi, 1]

One set of coordinate surfaces are oblate ellipsoids of rotation each one corresponding to a single
value Xxi (= η) with 0 ≤ Xxi < ∞, while $-\frac{\pi}{2}$ ≤ Eeta (= θ) ≤ $\frac{\pi}{2}$ and $-\pi < \psi \leq \pi$.
The equal semi-axes are longer than the third semi-axis || to the z-axis.
A single value of Eeta (= θ) $-\pi/2 \leq \theta \leq \pi/2$ gives a hyperboloid of rotation consisting of one sheet
as it is obtained by rotating a hyperbola around an axis passing through the middle between the
foci. The azimuth varies within $-\pi \leq \phi \leq \pi$ and Xxi (= η) in the interval 0≤η<∞. There is an infinite
number of such coordinate systems. Each system corresponds to a given value of the parameter
a=e determining the distance of the foci common to all conics of the system from the origin.
The default value of a is unity as shown above.

SetCoordinates[OblateSpheroidal[η, θ, ψ, e]]

OblateSpheroidal[η, θ, ψ, e]

CoordinatesToCartesian[{η,θ,ψ}]

{e Cos[θ] Cos[ψ] Cosh[η], e Cos[θ] Cosh[η] Sin[ψ], e Sin[θ] Sinh[η]}

CoordinateRanges[]

$$\left\{ 0 \leq \eta < \infty, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, -\pi < \psi \leq \pi \right\}$$

The parameter e may be any positive real number:

ParameterRanges[]

$0 < \#1 < \infty$

Bipolar coordinates u, v, z are a system corresponding to parallel wires, Moon Spencer Fig. 2.09; p.64 and p.51

SetCoordinates[Bipolar]

Bipolar[Uu, Vv, Zz, 1]

CoordinateRanges[]

$\{-\pi < Uu \le \pi,\ -\infty < Vv < \infty,\ -\infty < Zz < \infty\}$

Note: *Mathematica* uses the default names and coordinate ranges as just given.
We use the letters u and v just reversed to that use to follow Moon and Spencer's notation.

SetCoordinates[Bipolar[v, u, z, e]]

Bipolar[v, u, z, e]

CoordinateRanges[]

$\{-\pi < v \le \pi,\ -\infty < u < \infty,\ -\infty < z < \infty\}$

ParameterRanges[]

$0 < \#1 < \infty$

cuvz = CoordinatesToCartesian[{v,u,z}]

$$\left\{ \frac{e\,\text{Sinh}[u]}{-\text{Cos}[v] + \text{Cosh}[u]},\ \frac{e\,\text{Sin}[v]}{-\text{Cos}[v] + \text{Cosh}[u]},\ z \right\}$$

CoordinatesFromCartesian[{x,y,z}]

$$\left\{ -2\,\text{Im}\left[\text{ArcCoth}\left[\frac{x + \mathbb{i}\,y}{e}\right]\right],\ 2\,\text{Re}\left[\text{ArcCoth}\left[\frac{x + \mathbb{i}\,y}{e}\right]\right],\ z \right\}$$

CoordinatesToCartesian[{0.2 N[Pi/2],0.7 N[Pi/2], 0}]

$\{1.86213\,e,\ 0.431064\,e,\ 0\}$

CoordinatesFromCartesian[{1.86,.43,z}]

$$\left\{ -2\,\text{Im}\left[\text{ArcCoth}\left[\frac{1.86 + 0.43\,\mathbb{i}}{e}\right]\right],\ 2\,\text{Re}\left[\text{ArcCoth}\left[\frac{1.86 + 0.43\,\mathbb{i}}{e}\right]\right],\ z \right\}$$

Clear[x, y, z]
x[u_, v_] = cuvz[[1]]
y[u_, v_] = cuvz[[2]]

$$\frac{e\,\text{Sinh}[u]}{-\text{Cos}[v] + \text{Cosh}[u]}$$

$$\frac{e\,\text{Sin}[v]}{-\text{Cos}[v] + \text{Cosh}[u]}$$

u = -∞ (+∞) gives the poles x = -a = -2 (a = 2). u = const. are circles around these. Above
u ϵ [-2,2] with stepwidth Δu = 0.25, -π ≤ v ≤ π.

u = 0 is the y-axis. v = const, with u ϵ [-∞, ∞] gives circles passing through the poles.
Above v ϵ [-π, π] with Δv = π/6.
v = 0, ±π corresponds to points on the x-axis. v = π, -π: x ϵ [-1,1]; v = 0: x ϵ [-∞, -1] $\vee$ [1,∞].
The coordinate Zz extends these coordinates to a cylindrical system.

Bispherical coordinates η, θ, ψ, Moon Spencer Table E4Rx and Fig. 4.03, pp.110

are obtained from the plane bipolar coordinates by rotating them around the x-axis;
and renaming v -> η, u -> θ, ψ the new azimuthal coordinate:

X = y[θ, η] Cos[ψ]

Y = y[θ, η] Sin[ψ]

Z = x[θ, η]

SetCoordinates[Bispherical[]]

Bispherical[Uu, Vv, Pphi, 1]

SetCoordinates[Bispherical[θ, η, ψ, e]]

Bispherical[θ, η, ψ, e]

CoordinateRanges[]

$\{0 \le \theta \le \pi, \ -\infty < \eta < \infty, \ -\pi < \psi \le \pi\}$

ParameterRanges[]

$0 < \sharp 1 < \infty$

CoordinatesToCartesian[{θ, η, ψ}]

$$\left\{ \frac{e \, \text{Cos}[\psi] \, \text{Sin}[\theta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]}, \ \frac{e \, \text{Sin}[\theta] \, \text{Sin}[\psi]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]}, \ \frac{e \, \text{Sinh}[\eta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]} \right\}$$

CoordinatesFromCartesian[{x, y, z}]

$$\left\{ -2 \, \text{Im}\left[\text{ArcCoth}\left[\frac{\mathrm{i} \, \sqrt{x^2 + y^2} + z}{e} \right] \right], \ 2 \, \text{Re}\left[\text{ArcCoth}\left[\frac{\mathrm{i} \, \sqrt{x^2 + y^2} + z}{e} \right] \right], \ \text{ArcTan}[x, y] \right\}$$

Toroidal coordinates η, θ, ψ, Moon Spencer Table E4Ry and Fig 4.03, pp.112

are obtained from the plane bipolar coordinates by rotating them around the y-axis;
and renaming v -> η, u -> θ, ψ the new azimuthal coordinate:

X = x[η, θ] Sin[ψ]

$$\frac{e \, \text{Sin}[\psi] \, \text{Sinh}[\eta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]}$$

Y = x[η, θ] Sin[ψ]

$$\frac{e \, \text{Sin}[\psi] \, \text{Sinh}[\eta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]}$$

Z = y[η, θ]

$$\frac{e \, \text{Sin}[\theta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]}$$

```
SetCoordinates[Toroidal[]]
```

Toroidal[Uu, Vv, Pphi, 1]

```
SetCoordinates[Toroidal[θ, η, ψ, e]]
```

Toroidal[θ, η, ψ, e]

```
CoordinateRanges[]
```

$\{-\pi < \theta \le \pi, 0 \le \eta < \infty, -\pi < \psi \le \pi\}$

```
ParameterRanges[]
```

$0 < \#1 < \infty$

```
CoordinatesToCartesian[{θ, η, ψ}]
```

$$\left\{ \frac{e \, \text{Cos}[\psi] \, \text{Sinh}[\eta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]}, \frac{e \, \text{Sin}[\psi] \, \text{Sinh}[\eta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]}, \frac{e \, \text{Sin}[\theta]}{-\text{Cos}[\theta] + \text{Cosh}[\eta]} \right\}$$

```
CoordinatesFromCartesian[{x, y, z}]
```

$$\left\{ -2 \, \text{Im}\left[\text{ArcCoth}\left[\frac{\sqrt{x^2 + y^2} + i \, z}{e} \right] \right], 2 \, \text{Re}\left[\text{ArcCoth}\left[\frac{\sqrt{x^2 + y^2} + i \, z}{e} \right] \right], \text{ArcTan}[x, y] \right\}$$

Ellipsoidal coordinates λ, μ, ν are defined in Moon Spencer in Table 1.10, pp.40 f
The definitions of *Mathematica* do not agree with those of Moon-Spencer !

```
SetCoordinates[ConfocalEllipsoidal[]]
```

ConfocalEllipsoidal[Llambda, Mmu, Nnu, 3, 2, 1]

```
CoordinateRanges[]
```

$\{-\infty < \text{Llambda} < 1, 1 < \text{Mmu} < 4, 4 < \text{Nnu} < 9\}$

```
ParameterRanges[]
```

$0 < \#3 < \#2 < \#1 < \infty$

The coordinate surfaces are three sets of confocal conics: Three-axial ellipsoids, whose semi-axes have the lengths 3,2,1 in the default case. λ = const. determines a certain ellipsoid. λ may vary in the range -∞ < λ < 1. μ = const. (1 < μ < 4) gives a hyperboloid of one sheet. ν (4 < ν < 9) gives a hyperboloid of two sheets.
There are three parameter values determining the ratios of the semi-axes of the conics and the ranges of the coordinate values.

```
<< VectorAnalysis`
```

```
SetCoordinates[ConfocalEllipsoidal[λ, μ, ν, a, b, c]]
```

ConfocalEllipsoidal[λ, μ, ν, a, b, c]

```
CoordinateRanges[]
```

$\{-\infty < \lambda < c^2, c^2 < \mu < b^2, b^2 < \nu < a^2\}$

```
ParameterRanges[]
```

$0 < \#3 < \#2 < \#1 < \infty$

```
CoordinatesToCartesian[{λ,μ,ν}]
```

$$\left\{ \sqrt{\frac{\left(a^2 - \lambda\right)\,\left(a^2 - \mu\right)\,\left(a^2 - \nu\right)}{\left(a^2 - b^2\right)\,\left(a^2 - c^2\right)}} \,,\, \sqrt{\frac{\left(b^2 - \lambda\right)\,\left(b^2 - \mu\right)\,\left(b^2 - \nu\right)}{\left(-a^2 + b^2\right)\,\left(b^2 - c^2\right)}} \,,\, \sqrt{\frac{\left(c^2 - \lambda\right)\,\left(c^2 - \mu\right)\,\left(c^2 - \nu\right)}{\left(-a^2 + c^2\right)\,\left(-b^2 + c^2\right)}} \right\}$$

14.3.6 Arclength and Volume Element

The package "**VectorAnalysis`**" permits one to calculate differential arc length and the Jacobian matrix needed for volume integrals in the orthogonal curvilinear coordinates listed in the preceeding paragraph. In three-dimensional Cartesian coordinates a curve may be given by a parametric representation { x(t), y(t), z(t) }. Then ds, the differential of arc length s is given by the arc length factor $[\,x^2 + y^2 + z^2]^{1/2}$ multiplied with dt:

$$ds = [\,x^2 + y^2 + z^2]^{1/2} \; dt \;.$$

```
Clear[x,y,z]
SetCoordinates[Cartesian[x,y,z]]
```

Cartesian[x, y, z]

```
ArcLengthFactor[{x,y,z},t]
```

$$\sqrt{Dt[x, t]^2 + Dt[y, t]^2 + Dt[z, t]^2}$$

In cylindrical coordinates the arc length factor is for a curve given in parametric representation by r(t), φ(t), z(t) :

```
ArcLengthFactor[{r,φ,z},t,Cylindrical]
```

$$\sqrt{Dt[r, t]^2 + Dt[z, t]^2 + r^2\,Dt[\varphi, t]^2}$$

In spherical coordinates the arc length factor is

```
ArcLengthFactor[{r,θ,φ},t,Spherical]
```

$$\sqrt{Dt[r, t]^2 + r^2\,Dt[\theta, t]^2 + r^2\,Dt[\varphi, t]^2\,Sin[\theta]^2}$$

`ArcLengthFactor[{f1,f2,f3},t]`	give the derivative of the arc length along the curve {f1(t),f2(t),f3(t)} in the default coordinate system. f1, f2, f3 are the three independent variables of the system.
`ArcLengthFactor[{f1,f2,f3},t, `*`coordsys`*`]`	give the derivative in the system *coordsys*

With the help of the arc length factor it is easy to calculate the length of a curve given in parametric representation. One just integrates over the arc length factor for the range of the paramter belonging to the part of the curve, whose length is required. Note that the total derivative Dt (cf. § **10.1.2**) is used in the **ArcLengthFactor[]**. Therefore for any symbols their dependence on the paramter t is assumed. Any symbol representing a constant, e.g. ar, must be so declared by **Attributes[ar, Constant].**

Below a screw is given in Cartesian coordinates:

```
screw = {5 Cos[t], 5 Sin[t], 3 t};
```

```
Integrate[ ArcLengthFactor[screw, t, Cartesian], {t, 0, 2 π}]
```

$2 \sqrt{34}\; \pi$

Here a circle of radius 1 is given in cylindrical coordinates:

```
kreis = {1, t, 0};
```

```
Integrate[ ArcLengthFactor[kreis, t, Cylindrical], {t,0,2π}]
```

2π

Now the radius of the circle is a, i.e. a symbol. It represents the constant radius of the circle. As explained above this property must declarted by **Attributes[]** so that the calculation runs properly.

```
kreis = {a, t, 0} ;
SetAttributes[a, Constant]
```

```
Integrate[ ArcLengthFactor[kreis, t, Cylindrical], {t,0,2π}]
```

$2 \sqrt{a^2} \; \pi$

The scale factors h1, h2, h3 (called metric elements at the beginning of § **14.2.2**) are the square roots
of the diagonal elements of the diagonal metric tensor, Their product is the Jacobian determinant.

ScaleFactors[]	The 3 scale factors for the default coordinate system
ScaleFactors[*coordsys*]	The 3 scale factors for the system ***coordsys***

```
ScaleFactors[Cylindrical[r,φ,z]]
```

$\{1, r, 1\}$

```
ScaleFactors[Spherical]
```

$\{1, Rr, Rr \, Sin[Ttheta]\}$

The Jacobian matrix is the matrix of all the partial derivatives of the coordinate transformation
$x_i(u_1,u_2,u_3)$ w.r.t. the curvilinear coordinates u_k. Its determinant measures how the volume
element changes under this transformation. So the matrix is:

JacobianMatrix[]	Jacobian Matrix in default coordinate system
JacobianMatrix[*coordsys*]	Jacobian Matrix in default coordinate system ***coordsys***
JacobianDeterminant[]	Determinant of Jacobian Matrix

```
JacobianMatrix[Cylindrical[r,φ,z]]//MatrixForm
```

$$\begin{pmatrix} Cos[\varphi] & -r\,Sin[\varphi] & 0 \\ Sin[\varphi] & r\,Cos[\varphi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
JacobianDeterminant[Cylindrical]
```

Rr

```
SetCoordinates[Spherical[r, θ, φ]];
```

```
jm = JacobianMatrix[]; MatrixForm[jm]
```

$$\begin{pmatrix} Cos[\varphi]\,Sin[\theta] & r\,Cos[\theta]\,Cos[\varphi] & -r\,Sin[\theta]\,Sin[\varphi] \\ Sin[\theta]\,Sin[\varphi] & r\,Cos[\theta]\,Sin[\varphi] & r\,Cos[\varphi]\,Sin[\theta] \\ Cos[\theta] & -r\,Sin[\theta] & 0 \end{pmatrix}$$

JacobianDeterminant[]

$r^2 \, \text{Sin}[\theta]$

The volume of a sphere of radius a is:

Integrate[JacobianDeterminant[], {r,0,a},{θ,0,π},{φ,0,2π}]

$\dfrac{4 \, a^3 \, \pi}{3}$

14.4 Vector Field Solutions of the Vector Helmholtz Equation

The scalar Hemholtz equation is:

$$(\Delta + k^2) \Phi = 0.$$

The corresponding equation for vector fields **A** is the vector Helmholtz equation:

$$(\vec{\Delta} + k^2 \vec{I}) \mathbf{A} = (\text{graddiv} - \text{rotrot} + k^2 \vec{I}) \mathbf{A} = 0.$$

$\vec{I}$ is the unit matrix. In Cartesian coordinates this equation separates into three independent scalar Helmholtz equations, each for one component of the Cartesian components A_x, A_y, A_z.

However, in curvilinear coordinates the operator $\vec{\Delta}$ mixes the components. There are six coordinate systems in which a decomposition into three scalar field equations can be achieved. These are the four cylindrical systems (Cartesian, Cylindrical, EllipticCylindrical, ParabolicCylindrical) and the two systems with spherical coordinate surfaces (Spherical, Conical).

14.4.1 Cylindrical systems

In the cylindrical systems denoted in the Table 14.1 by **MC** (Cartesian, Cylindrical, EllipticCylindrical, ParabolicCylindrical the vector Helmholtz equation can be separated by the following vector fields

$$\mathbf{L} = \text{grad}(\psi) , \qquad \mathbf{M} = \text{curl}(\mathbf{ez}\ \psi), \qquad \mathbf{N} = 1/k\ \text{curl}\ \mathbf{M},$$

provided ψ is a solution of the scalar Helmholtz equation $\Delta\psi + k^2\psi = 0$ and **ez** is a constant vector along the cylindrical axis. With the same condition the vector fields **M** and **N** are solutions of Maxwell's eqs. and of the eq. "curl curl . - k^2 . = 0" , but the fields **L** are not.

Note: **L** is a solution of the homogeneous and the inhomogenous vector Helmholtz equation only, but not of the homogeneous Maxwell equations nor of the ensuing eq. "curl curl . - k^2 . = 0". **L** may be needed for solving the inhomogeneous Maxwell equations and the ensuing inhomogeneous equation curl curl . - k^2 . = ...

```
Needs["VectorAnalysis`"]
```

```
SetCoordinates[Cylindrical[r,φ,z]]
```

Cylindrical[r, φ, z]

```
a = {0,0,1} (* = ez *);
```

```
m = Curl[a ψ[r,φ,z]]
```

$$\left\{ \frac{\psi^{(0,1,0)}[\mathrm{r},\ \varphi,\ \mathrm{z}]}{\mathrm{r}},\ -\psi^{(1,0,0)}[\mathrm{r},\ \varphi,\ \mathrm{z}],\ 0 \right\}$$

```
Div[m]
```

0

Now it is shown that the vector field M = m is a solution of the eq. "curl curl . - k^2 . = 0". Then it is also a solution of the vector Helmholtz eq. in view of the preceding result . At first the definition of the field   **m** is inserted into the expression giving the vector **c** .

```
c = Expand[Curl[Curl[m]] - k^2 m]
```

$$\left\{ -\frac{k^2 \, \psi^{(0,1,0)}[r, \varphi, z]}{r} - \frac{\psi^{(0,1,2)}[r, \varphi, z]}{r} - \right.$$
$$\frac{\psi^{(0,3,0)}[r, \varphi, z]}{r^3} - \frac{\psi^{(1,1,0)}[r, \varphi, z]}{r^2} - \frac{\psi^{(2,1,0)}[r, \varphi, z]}{r},$$
$$-\frac{2 \, \psi^{(0,2,0)}[r, \varphi, z]}{r^3} + k^2 \, \psi^{(1,0,0)}[r, \varphi, z] - \frac{\psi^{(1,0,0)}[r, \varphi, z]}{r^2} + \psi^{(1,0,2)}[r, \varphi, z] +$$
$$\left. \frac{\psi^{(1,2,0)}[r, \varphi, z]}{r^2} + \frac{\psi^{(2,0,0)}[r, \varphi, z]}{r} + \psi^{(3,0,0)}[r, \varphi, z], \, 0 \right\}$$

```
c[[1]]//Simplify
```

$$-\frac{1}{r^3} \left(k^2 \, r^2 \, \psi^{(0,1,0)}[r, \varphi, z] + r^2 \, \psi^{(0,1,2)}[r, \varphi, z] + \right.$$
$$\left. \psi^{(0,3,0)}[r, \varphi, z] + r \, \psi^{(1,1,0)}[r, \varphi, z] + r^2 \, \psi^{(2,1,0)}[r, \varphi, z] \right)$$

Up to now we did not use the assumption that ψ is a solution of the scalar Helmholtz equation:

```
rps = Expand[Laplacian[ψ[r,φ,z]] + k^2 ψ[r,φ,z]] == 0
```

$$k^2 \, \psi[r, \varphi, z] + \psi^{(0,0,2)}[r, \varphi, z] + \frac{\psi^{(0,2,0)}[r, \varphi, z]}{r^2} + \frac{\psi^{(1,0,0)}[r, \varphi, z]}{r} + \psi^{(2,0,0)}[r, \varphi, z] == 0$$

We intend to solve this equation for the second derivative of ψ w.r.t. **z,** i.e. w.r.t. **D[ψ,{z,2}].** Before we can do that, we must find out how this command is stored in the list of commands of the last eq. This is displayed by the following command (cf. § **23.1**) :

```
FullForm[rps]
Equal[Plus[Times[Power[k, 2], \[Psi]][r,\[CurlyPhi],z]],
  Derivative[0, 0, 2][\[Psi]][r, \[CurlyPhi], z],
  Times[Power[r, -2], Derivative[0, 2, 0][\[Psi]][r, \[CurlyPhi], z]],
  Times[Power[r, -1], Derivative[1, 0, 0][\[Psi]][r, \[CurlyPhi], z]],
  Derivative[2, 0, 0][\[Psi]][r, \[CurlyPhi], z]], 0]
```

```
sps = ExpandAll[Solve[rps, Derivative[0,0,2][ψ][r,φ,z]]]//Flatten
```

$$\left\{ \psi^{(0,0,2)}[r, \varphi, z] \rightarrow \right.$$
$$\left. -k^2 \, \psi[r, \varphi, z] - \frac{\psi^{(0,2,0)}[r, \varphi, z]}{r^2} - \frac{\psi^{(1,0,0)}[r, \varphi, z]}{r} - \psi^{(2,0,0)}[r, \varphi, z] \right\}$$

Only the third derivatives of psi occur in the vector c. Therefore these are calculated and combined in a list **dsps** below.

```
drsps = Flatten[D[sps,r]]
```

$$\left\{ \psi^{(1,0,2)}[r, \varphi, z] \rightarrow \frac{2 \, \psi^{(0,2,0)}[r, \varphi, z]}{r^3} - k^2 \, \psi^{(1,0,0)}[r, \varphi, z] + \right.$$
$$\left. \frac{\psi^{(1,0,0)}[r, \varphi, z]}{r^2} - \frac{\psi^{(1,2,0)}[r, \varphi, z]}{r^2} - \frac{\psi^{(2,0,0)}[r, \varphi, z]}{r} - \psi^{(3,0,0)}[r, \varphi, z] \right\}$$

```
dpsps = Flatten[D[sps,φ]]
```

$$\left\{ \psi^{(0,1,2)}[r, \varphi, z] \rightarrow \right.$$
$$\left. -k^2 \, \psi^{(0,1,0)}[r, \varphi, z] - \frac{\psi^{(0,3,0)}[r, \varphi, z]}{r^2} - \frac{\psi^{(1,1,0)}[r, \varphi, z]}{r} - \psi^{(2,1,0)}[r, \varphi, z] \right\}$$

```
dzsps = Flatten[D[sps,z]]
```

$\left\{\psi^{(0,0,3)}\left[r,\ \varphi,\ z\right]\ \rightarrow\right.$

$\qquad -k^2\,\psi^{(0,0,1)}\left[r,\ \varphi,\ z\right]-\dfrac{\psi^{(0,2,1)}\left[r,\ \varphi,\ z\right]}{r^2}-\dfrac{\psi^{(1,0,1)}\left[r,\ \varphi,\ z\right]}{r}-\psi^{(2,0,1)}\left[r,\ \varphi,\ z\right]\left.\right\}$

```
dsps = Join[drsps, dpsps, dzsps];
```

Now these third derivatives are inserted into the first, second, third component of the vector **c** ; finally into the total vector in one step.

```
Expand[c[[1]] /. dsps]
```

0

```
Together[c[[2]] /. dsps]
```

0

```
Together[c[[3]] /. dsps]
```

0

```
Together[c /. dsps]
```

{0, 0, 0}

This completes the proof that the vector field **M = m** satisfies the equation **curl curl M - k^2 M = 0** and the vector Helmholtz equation. The same proof is now given for the vector field **N = n** .

```
n = Curl[Curl[a ψ[r,φ,z]]];      Div[n]
```

0

```
c = Expand[Curl[Curl[n]] - k^2 n ]
```

$\left\{\dfrac{2\,\psi^{(0,2,1)}\left[r,\ \varphi,\ z\right]}{r^3}-k^2\,\psi^{(1,0,1)}\left[r,\ \varphi,\ z\right]+\dfrac{\psi^{(1,0,1)}\left[r,\ \varphi,\ z\right]}{r^2}-\right.$

$\quad \psi^{(1,0,3)}\left[r,\ \varphi,\ z\right]-\dfrac{\psi^{(1,2,1)}\left[r,\ \varphi,\ z\right]}{r^2}-\dfrac{\psi^{(2,0,1)}\left[r,\ \varphi,\ z\right]}{r}-\psi^{(3,0,1)}\left[r,\ \varphi,\ z\right],$

$\quad -\dfrac{k^2\,\psi^{(0,1,1)}\left[r,\ \varphi,\ z\right]}{r}-\dfrac{\psi^{(0,1,3)}\left[r,\ \varphi,\ z\right]}{r}-\dfrac{\psi^{(0,3,1)}\left[r,\ \varphi,\ z\right]}{r^3}-\dfrac{\psi^{(1,1,1)}\left[r,\ \varphi,\ z\right]}{r^2}-$

$\quad \dfrac{\psi^{(2,1,1)}\left[r,\ \varphi,\ z\right]}{r},\ \dfrac{4\,\psi^{(0,2,0)}\left[r,\ \varphi,\ z\right]}{r^4}+\dfrac{k^2\,\psi^{(0,2,0)}\left[r,\ \varphi,\ z\right]}{r^2}+$

$\quad \dfrac{\psi^{(0,2,2)}\left[r,\ \varphi,\ z\right]}{r^2}+\dfrac{\psi^{(0,4,0)}\left[r,\ \varphi,\ z\right]}{r^4}+\dfrac{\psi^{(1,0,0)}\left[r,\ \varphi,\ z\right]}{r^3}+\dfrac{k^2\,\psi^{(1,0,0)}\left[r,\ \varphi,\ z\right]}{r}+$

$\quad \dfrac{\psi^{(1,0,2)}\left[r,\ \varphi,\ z\right]}{r}-\dfrac{2\,\psi^{(1,2,0)}\left[r,\ \varphi,\ z\right]}{r^3}+k^2\,\psi^{(2,0,0)}\left[r,\ \varphi,\ z\right]-\dfrac{\psi^{(2,0,0)}\left[r,\ \varphi,\ z\right]}{r^2}+$

$\quad \psi^{(2,0,2)}\left[r,\ \varphi,\ z\right]+\dfrac{2\,\psi^{(2,2,0)}\left[r,\ \varphi,\ z\right]}{r^2}+\dfrac{2\,\psi^{(3,0,0)}\left[r,\ \varphi,\ z\right]}{r}+\psi^{(4,0,0)}\left[r,\ \varphi,\ z\right]\left.\right\}$

Here the fourth and the third derivatives of $D[\psi, \{z,2\}]$ (ψ a solution of the scalar Helmholtz equation) must be inserted into the vector **c**.

```
ddsps = Flatten[Union[D[sps,r,r],D[sps,φ,φ],D[sps,z,z],
           D[sps,r,φ], D[sps,φ,z], D[sps,r,z], dsps]];
```

```
Together[c /. ddsps]
```

{0, 0, 0}

The field   **L = l** is a solution of the vector Helmholtz equation provided ψ is a solution of the scalar Helmholtz equation:

```
l = Grad[ ψ[r,φ,z] ]
```

$$\left\{ \psi^{(1,0,0)}[r, \varphi, z], \frac{\psi^{(0,1,0)}[r, \varphi, z]}{r}, \psi^{(0,0,1)}[r, \varphi, z] \right\}$$

```
Curl[l]
```

$\{0, 0, 0\}$

```
c = Grad[ Div[l] ] + k^2 l ;
```

```
Expand[c /. dsps ]
```

$\{0, 0, 0\}$

14.3.2 Systems with a radial vector

In the systems with spherical coordinate surfaces denoted in the Table 14.1 by   **MR** (Spherical, Conical)
the vector Helmholtz equation can be separated by the following vector fields :

   L = grad(ψ) ,   **M** = curl(**r** ψ),   **N** = 1/k curl **M.**

r is the radius vector. **M** and **N** only are solutions of Maxwells eqs. and of the eq. "curl curl . - k^2 . = 0" provided ψ is a solution of the scalar Helmholtz equation. **M** and **N** are solenoidal (source-free rotational) vectors fields. **L** gives irrotational source fields.
 The proof follows the same lines as that in the cylindrical case, but one must use the vector **r** = (r,0,0).
 See ex.14.8 .

14.5 Exercises

14.1 A plane contains the point C = (cx,cy,cz) and the vectors a = (ax,ay,az) and b = (bx,by,bz). Compute a general formula for the distance d of a point P = (px,py,pz) from this plane.

14.2 Check which of the following polynomials are solutions of the potential equation:
1. $f1 = (x + I\, y)^2 \, z^2$, 2. $f2 = (x + I\, y)^2 \, z$.

14.3 In the plane a triangle is given by the coordinates of its three corners. There is given a point P. Develop a test which checks whether P is inside, on the or outside the triangle.

Hints: H1: A half-ray emanating from P crosses a closed polygon in an odd number of points, if P is in the interior. H2: The vector to P may be spanned on two vectors forming the triangle. The expansion coefficients must be positive and their sum < 1, if P is inside the triangle. H3. Let a,b,c be the vectors pointing from P to the 3 corners. The vector products (a x b), (b x c), (c x a) must all be right-handed (or all left-handed) if the point is in the interior. One of these is zero if the point is on an edge.

14.4 The point P = (1,2,3) has the rectangular coordinates just given.
a) What are the values of its spherical coordinates ?
b) What are the values of its parabolic coordinates ?
c) What are the values of its coordinates in prolate spheroidal coordinates with focal length a = .5 ?

14.5 The point P has spherical coordinates $r = 3, \vartheta = 3\pi/7, \varphi = .3\,\pi$.
a) What are the values of its cylindrical coordinates ?
b) What are the values of its parabolic coordinates ?
c) What are the values of its coordinates in prolate spheroidal coordinates with focal length a = 2 ?

14.6 Compute the length of the arc of an Archimedian spiral $r = a\,\varphi$ extending from $\varphi = 0$ to $\varphi = \pi$.

14.7 Show that the vector fields L, M, N defined in sect. 14.3.1 are solutions of the vector Helmholtz equation in parabolic cylinder coordinates provided ψ is a solution of the scalar Helmholtz eq. in these coordinates.
a = constant unit vector along z-direction.

14.8 Show that the vector fields L, M, N defined in sect. 14.3.2 are solutions of the vector Helmholtz equation in spherical coordinates provided ψ is a solution of the scalar Helmholtz eq. in these coordinates. a = r = radial vector. In this case it is particularly important to insert anew all the previous solutions into the derivatives of $\partial^2\psi/\partial r^2$ -> ... .

B) VectorAnalysis using the kernel implementation

14.6 Coordinate systgems implemented in the kernel version

At the end I give a list of all the tutorials, guides and notebooks describing the comands implemented in the kernel version of vector analysis. The kernel version of vector analysis contains implementations for much more coordinate systems (65) than the package version (14 systems).

```
CoordinateChartData[All];
Length[%]
```

65

We confine this presentation to a selection, just the systems in 2 and 3 dimensions. For dimension 2 there are:

```
CoordinateChartData[{All, "Euclidean", 2}]
```

$\{\{\{\text{Bipolar}, \{\mathring{a}\}\}, \text{Euclidean}, 2\}, \{\text{Cartesian}, \text{Euclidean}, 2\},$
$\{\{\text{Confocal}, \{\mathring{\alpha}, \mathring{\beta}\}\}, \text{Euclidean}, 2\}, \{\{\text{Elliptic}, \{\mathring{a}\}\}, \text{Euclidean}, 2\},$
$\{\text{Polar}, \text{Euclidean}, 2\}, \{\text{PlanarParabolic}, \text{Euclidean}, 2\}\}$

2D
2.1 Bipolar   2.09, p.64
2.2 Cartesian   (1.01, p.9)

The letters with a point above and below stand for the parameters characterising the variants of each system. For example, in plane elliptic coordinates there are infinitely many systems, each one characterized by a particular value of the focal length $\overset{.}{\underset{.}{a}}$. Figure numbers and pages refer to Moon and Spencer.

CoordinateChartData[{All, "Euclidean", 3}]

```
{{{BipolarCylindrical, {a̤}}, Euclidean, 3},
 {{Bispherical, {a̤}}, Euclidean, 3}, {Cartesian, Euclidean, 3},
 {CircularParabolic, Euclidean, 3}, {{Confocal, {α̤, β̤, γ̤}}, Euclidean, 3},
 {{ConfocalParaboloidal, {α̤, β̤}}, Euclidean, 3}, {{Conical, {b̤, c̤}}, Euclidean, 3},
 {Cylindrical, Euclidean, 3}, {{EllipticCylindrical, {a̤}}, Euclidean, 3},
 {Hyperspherical, Euclidean, 3}, {{OblateSpheroidal, {a̤}}, Euclidean, 3},
 {ParabolicCylindrical, Euclidean, 3}, {{ProlateSpheroidal, {a̤}}, Euclidean, 3},
 {Spherical, Euclidean, 3}, {{Toroidal, {a̤}}, Euclidean, 3}}
```

3D

3.1	BipolarCylindrical	(2.09, p.64)
3.2	Bispherical	4.03, p.110
3.3	Cartesian	1.01, p.9
3.4	CircularParabolic	1.07, p.34
3.5	Confocal	1.10, p.41
3.6	ConfocalParaboloidal	1.11, p.44
3.7	Conical	1.09, p.37
3.8	Cylindrical	1.02, p.12
3.9	EllipticCylindrical	1.03, p.17
3.10	Hyperspherical	
3.11	OblateSpheroidal	1.07, p.31
3.12	Parabolic Cylindrical	1.04, p.21
3.13	Prolate Spheroidal	1.06, p.28
3.14	Spherical	1.05, p.24
3.15	Toroidal	4.04, p.112

The only default coordinate systems are the Cartesian ones. There particular dimensions may be chosen either explicitely or implicitely by the length of the list of variables. Other curvilinear systems never can be chosen as default systems. So in each command the name of the coordinate system and the list of variables must be given. Transformations from any coordinate system to any other system with the same dimensions may be accomplished with the commands

CoordinateTransformData[{*oldsys* -> *newsys*, *dim*}, "Mapping"]   **or**

CoordinateTransformData[{*oldsys* -> *newsys*, "Euclidean", *dim*}, "Mapping"]

See below for specific examples. The whole set of transformation pairs in 2 and 3 dimensions is given in the following list:

CoordinateTransformData[] // Length

278

The command
CoordinateTransformData[]
can be used to get the metric ("Metric"), the scalefactors ("ScaleFactors"),
the range of each coordinate ("CoordinateRangeAssumptions"),

the Jacobian matrix and determinant ("MappingJacobianDeterminant").

```
CoordinateTransformData["Properties"]
```

{InverseMappingJacobian, Mapping, MappingJacobian,
 MappingJacobianDeterminant, OrthonormalBasisRotation, StandardName}

14.7 Some examples for 2-dimensional coordinates

B2.2   Cartesian   {"Cartesian","Euclidean",2}   or   "Cartesian"

```
CoordinateChartData["Cartesian", "ScaleFactors", {x, y}]
```

{1, 1}

```
CoordinateChartData[{"Cartesian", "Euclidean", 2}, "ScaleFactors", {x, y}]
```

{1, 1}

```
CoordinateChartData[{"Cartesian", 2}, "CoordinateRangeAssumptions", {x, y}]
```

$x \in \text{Reals} \,\&\&\, y \in \text{Reals}$

```
Grad[Ψ[x, y], {x, y}, {"Cartesian", "Euclidean", 2}]
```

$\left\{ \Psi^{(1,0)}[x, y], \Psi^{(0,1)}[x, y] \right\}$

```
Laplacian[Ψ[x, y], {x, y}, {"Cartesian", "Euclidean", 2}]
```

$\Psi^{(0,2)}[x, y] + \Psi^{(2,0)}[x, y]$

```
va = {ax[x, y], ay[x, y]}
```

{ax[x, y], ay[x, y]}

```
Div[va, {x, y}, {"Cartesian", "Euclidean", 2}]
```

$ay^{(0,1)}[x, y] + ax^{(1,0)}[x, y]$

In two dimensions the curl of a vector field is a pseudoscalar?

```
Curl[va, {x, y}, {"Cartesian", "Euclidean", 2}]
```

$-ax^{(0,1)}[x, y] + ay^{(1,0)}[x, y]$

```
Laplacian[va, {x, y}, {"Cartesian", "Euclidean", 2}]
```

$\left\{ ax^{(0,2)}[x, y] + ax^{(2,0)}[x, y], \, ay^{(0,2)}[x, y] + ay^{(2,0)}[x, y] \right\}$

B2.5   Polar   {Polar,Euclidean,2}   or   "Polar"

```
CoordinateChartData["Polar", "Metric", {ρ, ϕ}] // MatrixForm
```

$\begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$

```
CoordinateChartData["Polar", "ScaleFactors", {ρ, ϕ}]
```

{1, ρ}

```
CoordinateChartData["Polar", "CoordinateRangeAssumptions", {ρ, ϕ}]
```

$\rho > 0 \,\&\&\, -\pi < \phi \leq \pi$

```
Grad[Ψ[ρ, ϕ], {ρ, ϕ}, "Polar"]
```

$\left\{ \Psi^{(1,0)}[\rho, \phi], \frac{\Psi^{(0,1)}[\rho, \phi]}{\rho} \right\}$

```
Laplacian[Ψ[ρ, ϕ], {ρ, ϕ}, "Polar"] // Expand
```

$\frac{\Psi^{(0,2)}[\rho, \phi]}{\rho^2} + \frac{\Psi^{(1,0)}[\rho, \phi]}{\rho} + \Psi^{(2,0)}[\rho, \phi]$

```
va = {aρ[ρ, ϕ], aϕ[ρ, ϕ]}
```

$\{ a\rho[\rho, \phi], a\phi[\rho, \phi] \}$

```
Div[va, {ρ, ϕ}, "Polar"] // Expand
```

$\frac{a\rho[\rho, \phi]}{\rho} + \frac{a\phi^{(0,1)}[\rho, \phi]}{\rho} + a\rho^{(1,0)}[\rho, \phi]$

```
Curl[va, {ρ, ϕ}, "Polar"] // Expand
```

$\frac{a\phi[\rho, \phi]}{\rho} - \frac{a\rho^{(0,1)}[\rho, \phi]}{\rho} + a\phi^{(1,0)}[\rho, \phi]$

```
Laplacian[va, {ρ, ϕ}, "Polar"] // Expand
```

$\left\{ -\frac{a\rho[\rho, \phi]}{\rho^2} - \frac{2\, a\phi^{(0,1)}[\rho, \phi]}{\rho^2} + \frac{a\rho^{(0,2)}[\rho, \phi]}{\rho^2} + \frac{a\rho^{(1,0)}[\rho, \phi]}{\rho} + a\rho^{(2,0)}[\rho, \phi], \right.$

$\left. -\frac{a\phi[\rho, \phi]}{\rho^2} + \frac{2\, a\rho^{(0,1)}[\rho, \phi]}{\rho^2} + \frac{a\phi^{(0,2)}[\rho, \phi]}{\rho^2} + \frac{a\phi^{(1,0)}[\rho, \phi]}{\rho} + a\phi^{(2,0)}[\rho, \phi] \right\}$

```
{xq, yq} = CoordinateTransformData["Polar" → "Cartesian", "Mapping", {ρ, ϕ}]
```

$\{ \rho \, \text{Cos}[\phi], \rho \, \text{Sin}[\phi] \}$

```
{ρq, ϕq} = CoordinateTransformData["Cartesian" → "Polar", "Mapping", {x, y}]
```

$\left\{ \sqrt{x^2 + y^2}, \text{ArcTan}[x, y] \right\}$

B2.6   Elliptic   "Elliptic"

There is an infinite number of plane elliptic coordinate systems; each one is characterized by a particular value of the excentricity e = å .

```
sue = å → e (* = a =  excentricity *)
```

å → e

```
CoordinateChartData["Elliptic", "Metric", {η, ψ}] // MatrixForm
```

$\begin{pmatrix} \frac{1}{2} \, å^2 \, (-\text{Cos}[2\,\psi] + \text{Cosh}[2\,\eta]) & 0 \\ 0 & \frac{1}{2} \, å^2 \, (-\text{Cos}[2\,\psi] + \text{Cosh}[2\,\eta]) \end{pmatrix}$

```
CoordinateChartData["Elliptic", "ScaleFactors", {η, ψ}] /. sue
```

$\left\{ \frac{e \, \sqrt{-\text{Cos}[2\,\psi] + \text{Cosh}[2\,\eta]}}{\sqrt{2}}, \frac{e \, \sqrt{-\text{Cos}[2\,\psi] + \text{Cosh}[2\,\eta]}}{\sqrt{2}} \right\}$

```
Map[Factor,
 Map[TrigExpand, %, {3}] /. {Sin[ψ]² → 1 - Cos[ψ]², Sinh[η]² → Cosh[η]² - 1 }, {2}]
```

$$\left\{ e \sqrt{-\text{Cos}[\psi]^2 + \text{Cosh}[\eta]^2} \, , \, e \sqrt{-\text{Cos}[\psi]^2 + \text{Cosh}[\eta]^2} \right\}$$

```
CoordinateChartData["Elliptic", "CoordinateRangeAssumptions", {η, ψ}]
```

$\eta > 0 \,\&\&\, -\pi < \psi \leq \pi \,\&\&\, (\eta \neq 0 \,||\, \psi \neq 0)$

```
Grad[Ψ[η, ψ], {η, ψ}, "Elliptic"] /. sue
```

$$\left\{ \frac{\sqrt{2} \; \Psi^{(1,0)}[\eta, \psi]}{e \sqrt{-\text{Cos}[2\psi] + \text{Cosh}[2\eta]}} , \frac{\sqrt{2} \; \Psi^{(0,1)}[\eta, \psi]}{e \sqrt{-\text{Cos}[2\psi] + \text{Cosh}[2\eta]}} \right\}$$

```
Laplacian[Ψ[η, ψ], {η, ψ}, "Elliptic"] /. sue // Expand // Simplify
```

$$- \frac{2 \left(\Psi^{(0,2)}[\eta, \psi] + \Psi^{(2,0)}[\eta, \psi] \right)}{e^2 \left(\text{Cos}[2\psi] - \text{Cosh}[2\eta] \right)}$$

```
va = {aη[η, ψ], aψ[η, ψ]}
```

$\{a\eta[\eta, \psi], a\psi[\eta, \psi]\}$

```
Div[va, {η, ψ}, "Elliptic"] /. sue // Expand // Simplify
```

$$\left(\sqrt{2} \left(a\psi[\eta, \psi] \, \text{Sin}[2\psi] + a\eta[\eta, \psi] \, \text{Sinh}[2\eta] - (\text{Cos}[2\psi] - \text{Cosh}[2\eta]) \right. \right.$$
$$\left. \left. \left(a\psi^{(0,1)}[\eta, \psi] + a\eta^{(1,0)}[\eta, \psi] \right) \right) \right) \Big/ \left(e \, (-\text{Cos}[2\psi] + \text{Cosh}[2\eta])^{3/2} \right)$$

```
Curl[va, {η, ψ}, "Elliptic"] /. sue // Expand // Simplify
```

$$- \left(\left(\left(\sqrt{2} \left(a\eta[\eta, \psi] \, \text{Sin}[2\psi] - a\psi[\eta, \psi] \, \text{Sinh}[2\eta] - (\text{Cos}[2\psi] - \text{Cosh}[2\eta]) \right. \right. \right. \right.$$
$$\left. \left. \left. \left(a\eta^{(0,1)}[\eta, \psi] - a\psi^{(1,0)}[\eta, \psi] \right) \right) \right) \Big/ \left(e \, (-\text{Cos}[2\psi] + \text{Cosh}[2\eta])^{3/2} \right) \right)$$

```
Laplacian[va, {η, ψ}, "Elliptic"] /. sue // Expand // Simplify
```

$$\left\{ - \left(\left(2 \left(a\eta[\eta, \psi] \, (\text{Cos}[2\psi] + \text{Cosh}[2\eta]) + 2 \, \text{Sinh}[2\eta] \, a\psi^{(0,1)}[\eta, \psi] + \text{Cos}[2\psi] \right. \right. \right. \right.$$
$$a\eta^{(0,2)}[\eta, \psi] - \text{Cosh}[2\eta] \, a\eta^{(0,2)}[\eta, \psi] - 2 \, \text{Sin}[2\psi] \, a\psi^{(1,0)}[\eta, \psi] + \text{Cos}[2\psi]$$
$$\left. \left. a\eta^{(2,0)}[\eta, \psi] - \text{Cosh}[2\eta] \, a\eta^{(2,0)}[\eta, \psi] \right) \right) \Big/ \left(e^2 \, (\text{Cos}[2\psi] - \text{Cosh}[2\eta])^2 \right) \right),$$
$$- \left(\left(2 \left(a\psi[\eta, \psi] \, (\text{Cos}[2\psi] + \text{Cosh}[2\eta]) - 2 \, \text{Sinh}[2\eta] \, a\eta^{(0,1)}[\eta, \psi] + \text{Cos}[2\psi] \right. \right. \right.$$
$$a\psi^{(0,2)}[\eta, \psi] - \text{Cosh}[2\eta] \, a\psi^{(0,2)}[\eta, \psi] + 2 \, \text{Sin}[2\psi] \, a\eta^{(1,0)}[\eta, \psi] + \text{Cos}[2\psi]$$
$$\left. \left. \left. a\psi^{(2,0)}[\eta, \psi] - \text{Cosh}[2\eta] \, a\psi^{(2,0)}[\eta, \psi] \right) \right) \Big/ \left(e^2 \, (\text{Cos}[2\psi] - \text{Cosh}[2\eta])^2 \right) \right) \right\}$$

```
Clear[x, y, η, ψ]
```

```
{xp, yp} =
 CoordinateTransformData["Elliptic" → "Cartesian", "Mapping", {η, ψ}] /. sue
```

$\{e \, \text{Cos}[\psi] \, \text{Cosh}[\eta], e \, \text{Sin}[\psi] \, \text{Sinh}[\eta]\}$

```
{ηp, ψp} =
 CoordinateTransformData["Cartesian" → "Elliptic", "Mapping", {x, y}] /. sue
```

$$\left\{\text{ArcCosh}\left[\frac{\sqrt{e^2 + x^2 + y^2 + \sqrt{-4 e^2 x^2 + (e^2 + x^2 + y^2)^2}}}{\sqrt{2}\ e}\right],\right.$$

$$\text{ArcTan}\left[x \sqrt{-2 e + \sqrt{2} \sqrt{e^2 + x^2 + y^2 + \sqrt{-4 e^2 x^2 + (e^2 + x^2 + y^2)^2}}}\right.$$

$$\sqrt{2 e + \sqrt{2} \sqrt{e^2 + x^2 + y^2 + \sqrt{-4 e^2 x^2 + (e^2 + x^2 + y^2)^2}}}\ ,$$

$$\left.\left.\sqrt{2}\ y \sqrt{e^2 + x^2 + y^2 + \sqrt{-4 e^2 x^2 + (e^2 + x^2 + y^2)^2}}\ \right]\right\}$$

```
CoordinateTransformData["Elliptic" → "Polar", "Mapping", {η, ψ}] /. sue
```

$$\left\{\sqrt{e^2 \cos[\psi]^2 \cosh[\eta]^2 + e^2 \sin[\psi]^2 \sinh[\eta]^2}\ ,\right.$$
$$\left.\text{ArcTan}[e \cos[\psi] \cosh[\eta], e \sin[\psi] \sinh[\eta]]\right\}$$

```
CoordinateTransformData["Polar" → "Elliptic", "Mapping", {ρ, ϕ}] /. sue
```

$$\left\{\text{ArcCosh}\left[\frac{1}{\sqrt{2}\ e}\right.\right.$$

$$\left(\sqrt{\left(e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2 + \sqrt{-4 e^2 \rho^2 \cos[\phi]^2 + (e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2)^2}\right)}\right)\right],$$

$$\text{ArcTan}\left[\rho \cos[\phi] \sqrt{\left(-2 e + \sqrt{2} \sqrt{\left(e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2 + \right.}\right.}\right.$$
$$\left.\left.\sqrt{\left(-4 e^2 \rho^2 \cos[\phi]^2 + (e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2)^2\right)}\right)\right)$$
$$\sqrt{\left(2 e + \sqrt{2} \sqrt{\left(e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2 + \sqrt{\left(-4 e^2 \rho^2 \cos[\phi]^2 + \right.}\right.}\right.}$$
$$\left.\left.\left.(e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2)^2\right)\right)\right), \sqrt{2}\ \rho \sin[\phi]$$

$$\left.\left.\sqrt{\left(e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2 + \sqrt{-4 e^2 \rho^2 \cos[\phi]^2 + (e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2)^2}\right)}\right]\right\}$$

The transformation above may also be done in two steps with Cartesian coordinates xp, yp as an intermedian step

```
{xp, yp} = CoordinateTransformData["Polar" → "Cartesian", "Mapping", {ρ, ϕ}]
```

$$\{\rho \cos[\phi], \rho \sin[\phi]\}$$

```
CoordinateTransformData["Cartesian" → "Elliptic", "Mapping", {xp, yp}] /. sue
```

$$\Big\{ \text{ArcCosh}\Big[\frac{1}{\sqrt{2}\ e}$$

$$\Big(\sqrt{\Big(e^2 + \rho^2 \cos[\phi]^2 + \rho^2 \sin[\phi]^2 + \sqrt{-4\,e^2\,\rho^2\,\cos[\phi]^2 + \big(e^2 + \rho^2\,\cos[\phi]^2 + \rho^2\,\sin[\phi]^2\big)^2}\,\Big)}\,\Big) \Big],$$

$$\text{ArcTan}\Big[\rho\,\cos[\phi]\,\sqrt{\Big(-2\,e + \sqrt{2}\,\sqrt{e^2 + \rho^2\,\cos[\phi]^2 + \rho^2\,\sin[\phi]^2 + }}$$

$$\sqrt{\big(-4\,e^2\,\rho^2\,\cos[\phi]^2 + \big(e^2 + \rho^2\,\cos[\phi]^2 + \rho^2\,\sin[\phi]^2\big)^2\big)}\,\Big)\Big)}$$

$$\sqrt{\Big(2\,e + \sqrt{2}\,\sqrt{\Big(e^2 + \rho^2\,\cos[\phi]^2 + \rho^2\,\sin[\phi]^2 + \sqrt{\big(-4\,e^2\,\rho^2\,\cos[\phi]^2 + }}}$$

$$\big(e^2 + \rho^2\,\cos[\phi]^2 + \rho^2\,\sin[\phi]^2\big)^2\big)\Big)\Big)}\,,\ \sqrt{2}\,\rho\,\sin[\phi]$$

$$\sqrt{\Big(e^2 + \rho^2\,\cos[\phi]^2 + \rho^2\,\sin[\phi]^2 + \sqrt{-4\,e^2\,\rho^2\,\cos[\phi]^2 + \big(e^2 + \rho^2\,\cos[\phi]^2 + \rho^2\,\sin[\phi]^2\big)^2}\,\Big)}\,\Big]\Big\}$$

14.8 Some examples for 3-dimensional coordinates

B3.3   Cartesian

```
CoordinateChartData["Cartesian", "Metric", {x, y, z}] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
CoordinateChartData["Cartesian", "ScaleFactors", {x, y, z}]
```

$\{1, 1, 1\}$

```
Grad[Ψ[x, y, z], {x, y, z}, "Cartesian"]
```

$\big\{ \Psi^{(1,0,0)}[x, y, z],\ \Psi^{(0,1,0)}[x, y, z],\ \Psi^{(0,0,1)}[x, y, z] \big\}$

```
Laplacian[Ψ[x, y, z], {x, y, z}, "Cartesian"] // Expand
```

$\Psi^{(0,0,2)}[x, y, z] + \Psi^{(0,2,0)}[x, y, z] + \Psi^{(2,0,0)}[x, y, z]$

```
va = {ax[x, y, z], ay[x, y, z], az[x, y, z]}
```

$\{ax[x, y, z], ay[x, y, z], az[x, y, z]\}$

```
Curl[va, {x, y, z}, "Cartesian"] // Expand
```

$\big\{ -ay^{(0,0,1)}[x, y, z] + az^{(0,1,0)}[x, y, z],$
$ax^{(0,0,1)}[x, y, z] - az^{(1,0,0)}[x, y, z],\ -ax^{(0,1,0)}[x, y, z] + ay^{(1,0,0)}[x, y, z] \big\}$

```
Laplacian[va, {x, y, z}, "Cartesian"] // Expand
```

$\big\{ ax^{(0,0,2)}[x, y, z] + ax^{(0,2,0)}[x, y, z] + ax^{(2,0,0)}[x, y, z],$
$ay^{(0,0,2)}[x, y, z] + ay^{(0,2,0)}[x, y, z] + ay^{(2,0,0)}[x, y, z],$
$az^{(0,0,2)}[x, y, z] + az^{(0,2,0)}[x, y, z] + az^{(2,0,0)}[x, y, z] \big\}$

B3.14 Spherical   1.05, p.24

```
CoordinateChartData["Spherical", "Metric", {r, θ, ψ}] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \, \text{Sin}[\theta]^2 \end{pmatrix}$$

```
CoordinateChartData["Spherical", "ScaleFactors", {r, θ, ψ}]
```

$\{1, r, r \, \text{Sin}[\theta]\}$

```
Grad[Ψ[r, θ, ψ], {r, θ, ψ}, "Spherical"]
```

$$\left\{ \Psi^{(1,0,0)}[r, \theta, \psi], \; \frac{\Psi^{(0,1,0)}[r, \theta, \psi]}{r}, \; \frac{\text{Csc}[\theta] \, \Psi^{(0,0,1)}[r, \theta, \psi]}{r} \right\}$$

```
Laplacian[Ψ[r, θ, ψ], {r, θ, ψ}, "Spherical"] // Expand
```

$$\frac{\text{Csc}[\theta]^2 \, \Psi^{(0,0,2)}[r, \theta, \psi]}{r^2} + \frac{\text{Cot}[\theta] \, \Psi^{(0,1,0)}[r, \theta, \psi]}{r^2} +$$

$$\frac{\Psi^{(0,2,0)}[r, \theta, \psi]}{r^2} + \frac{2 \, \Psi^{(1,0,0)}[r, \theta, \psi]}{r} + \Psi^{(2,0,0)}[r, \theta, \psi]$$

```
va = {ar[r, θ, ψ], aθ[r, θ, ψ], aψ[r, θ, ψ]}
```

$\{\text{ar}[r, \theta, \psi], \text{a}\theta[r, \theta, \psi], \text{a}\psi[r, \theta, \psi]\}$

```
Curl[va, {r, θ, ψ}, "Spherical"] // Expand
```

$$\left\{ \frac{\text{a}\psi[r, \theta, \psi] \, \text{Cot}[\theta]}{r} - \frac{\text{Csc}[\theta] \, \text{a}\theta^{(0,0,1)}[r, \theta, \psi]}{r} + \frac{\text{a}\psi^{(0,1,0)}[r, \theta, \psi]}{r}, \right.$$

$$-\frac{\text{a}\psi[r, \theta, \psi]}{r} + \frac{\text{Csc}[\theta] \, \text{ar}^{(0,0,1)}[r, \theta, \psi]}{r} - \text{a}\psi^{(1,0,0)}[r, \theta, \psi],$$

$$\left. \frac{\text{a}\theta[r, \theta, \psi]}{r} - \frac{\text{ar}^{(0,1,0)}[r, \theta, \psi]}{r} + \text{a}\theta^{(1,0,0)}[r, \theta, \psi] \right\}$$

```
Laplacian[va, {r, θ, ψ}, "Spherical"] // Expand
```

$$\Big\{ -\frac{2\, \text{ar}[r, \theta, \psi]}{r^2} - \frac{2\, \text{a}\theta[r, \theta, \psi]\, \text{Cot}[\theta]}{r^2} - \frac{2\, \text{Csc}[\theta]\, \text{a}\psi^{(0,0,1)}[r, \theta, \psi]}{r^2} +$$

$$\frac{\text{Csc}[\theta]^2\, \text{ar}^{(0,0,2)}[r, \theta, \psi]}{r^2} + \frac{\text{Cot}[\theta]\, \text{ar}^{(0,1,0)}[r, \theta, \psi]}{r^2} - \frac{2\, \text{a}\theta^{(0,1,0)}[r, \theta, \psi]}{r^2} +$$

$$\frac{\text{ar}^{(0,2,0)}[r, \theta, \psi]}{r^2} + \frac{2\, \text{ar}^{(1,0,0)}[r, \theta, \psi]}{r} + \text{ar}^{(2,0,0)}[r, \theta, \psi],$$

$$-\frac{\text{a}\theta[r, \theta, \psi]}{r^2} - \frac{\text{a}\theta[r, \theta, \psi]\, \text{Cot}[\theta]^2}{r^2} - \frac{2\, \text{Cot}[\theta]\, \text{Csc}[\theta]\, \text{a}\psi^{(0,0,1)}[r, \theta, \psi]}{r^2} +$$

$$\frac{\text{Csc}[\theta]^2\, \text{a}\theta^{(0,0,2)}[r, \theta, \psi]}{r^2} + \frac{2\, \text{ar}^{(0,1,0)}[r, \theta, \psi]}{r^2} + \frac{\text{Cot}[\theta]\, \text{a}\theta^{(0,1,0)}[r, \theta, \psi]}{r^2} +$$

$$\frac{\text{a}\theta^{(0,2,0)}[r, \theta, \psi]}{r^2} + \frac{2\, \text{a}\theta^{(1,0,0)}[r, \theta, \psi]}{r} + \text{a}\theta^{(2,0,0)}[r, \theta, \psi],$$

$$-\frac{\text{a}\psi[r, \theta, \psi]}{r^2} - \frac{\text{a}\psi[r, \theta, \psi]\, \text{Cot}[\theta]^2}{r^2} + \frac{2\, \text{Csc}[\theta]\, \text{ar}^{(0,0,1)}[r, \theta, \psi]}{r^2} +$$

$$\frac{2\, \text{Cot}[\theta]\, \text{Csc}[\theta]\, \text{a}\theta^{(0,0,1)}[r, \theta, \psi]}{r^2} + \frac{\text{Csc}[\theta]^2\, \text{a}\psi^{(0,0,2)}[r, \theta, \psi]}{r^2} +$$

$$\frac{\text{Cot}[\theta]\, \text{a}\psi^{(0,1,0)}[r, \theta, \psi]}{r^2} + \frac{\text{a}\psi^{(0,2,0)}[r, \theta, \psi]}{r^2} + \frac{2\, \text{a}\psi^{(1,0,0)}[r, \theta, \psi]}{r} + \text{a}\psi^{(2,0,0)}[r, \theta, \psi] \Big\}$$

```
{xp, yp, zp} =
 CoordinateTransformData["Spherical" → "Cartesian", "Mapping", {r, θ, ψ}]
```

$\{r\, \text{Cos}[\psi]\, \text{Sin}[\theta], r\, \text{Sin}[\theta]\, \text{Sin}[\psi], r\, \text{Cos}[\theta]\}$

```
{rp, θp, ψp} =
 CoordinateTransformData["Cartesian" → "Spherical", "Mapping", {x, y, z}]
```

$\Big\{ \sqrt{x^2 + y^2 + z^2}, \text{ArcTan}\big[z, \sqrt{x^2 + y^2}\big], \text{ArcTan}[x, y] \Big\}$

B3.13 Prolate Spheroidal

Here the excentricity of all the conics is denoted as a in accordance with Fig.1.06 of ref.[MS].

```
CoordinateChartData[{{"ProlateSpheroidal", {a}}, "Euclidean", 3},
  "Metric", {η, θ, ψ}] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\, \text{a}^2\, (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]) & 0 & 0 \\ 0 & \frac{1}{2}\, \text{a}^2\, (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]) & 0 \\ 0 & 0 & \text{a}^2\, \text{Sin}[\theta]^2\, \text{Sinh}[\eta]^2 \end{pmatrix}$$

```
CoordinateChartData[
 {{"ProlateSpheroidal", {a}}, "Euclidean", 3}, "ScaleFactors", {η, θ, ψ}]
```

$\Big\{ \frac{\text{a}\, \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]}}{\sqrt{2}}, \frac{\text{a}\, \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]}}{\sqrt{2}}, \text{a}\, \text{Sin}[\theta]\, \text{Sinh}[\eta] \Big\}$

```
Map[Factor,
 Map[TrigExpand, %, {3}] /. {Sin[θ]² → 1 - Cos[θ]², Sinh[η]² → Cosh[η]² - 1}, {2}]
```

$\Big\{ \text{a}\, \sqrt{-\text{Cos}[\theta]^2 + \text{Cosh}[\eta]^2}, \text{a}\, \sqrt{-\text{Cos}[\theta]^2 + \text{Cosh}[\eta]^2}, \text{a}\, \text{Sin}[\theta]\, \text{Sinh}[\eta] \Big\}$

Grad[Ψ[η, θ, ψ], {η, θ, ψ}, {{"ProlateSpheroidal", {a}}, "Euclidean", 3}]

$$\left\{ \frac{\sqrt{2}\ \Psi^{(1,0,0)}[\eta, \theta, \psi]}{a\ \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]}}, \right.$$

$$\left. \frac{\sqrt{2}\ \Psi^{(0,1,0)}[\eta, \theta, \psi]}{a\ \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]}}, \frac{\text{Csc}[\theta]\ \text{Csch}[\eta]\ \Psi^{(0,0,1)}[\eta, \theta, \psi]}{a} \right\}$$

Laplacian[Ψ[η, θ, ψ], {η, θ, ψ},
  {{"ProlateSpheroidal", {a}}, "Euclidean", 3}] // Expand // Simplify

$$\left((\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ \text{Csc}[\theta]^2\ \text{Csch}[\eta]^2\ \Psi^{(0,0,2)}[\eta, \theta, \psi] - \right.$$
$$2\ \big(\text{Cot}[\theta]\ \Psi^{(0,1,0)}[\eta, \theta, \psi] + \Psi^{(0,2,0)}[\eta, \theta, \psi] + \text{Coth}[\eta]\ \Psi^{(1,0,0)}[\eta, \theta, \psi] +$$
$$\left. \Psi^{(2,0,0)}[\eta, \theta, \psi] \big) \right) \big/ \left(a^2\ (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta]) \right)$$

va = {aη[η, θ, ψ], aθ[η, θ, ψ], aψ[η, θ, ψ]}

{aη[η, θ, ψ], aθ[η, θ, ψ], aψ[η, θ, ψ]}

Curl[va, {η, θ, ψ}, {{"ProlateSpheroidal", {a}}, "Euclidean", 3}] // Expand //
 Simplify

$$\left\{ \left(\sqrt{2}\ a\psi[\eta, \theta, \psi]\ \text{Cot}[\theta] - \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]}\ \text{Csc}[\theta]\ \text{Csch}[\eta]\ a\theta^{(0,0,1)}[\eta, \theta, \psi] + \right. \right.$$
$$\left. \sqrt{2}\ a\psi^{(0,1,0)}[\eta, \theta, \psi] \right) \big/ \left(a\ \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]} \right),$$
$$\left(-\sqrt{2}\ a\psi[\eta, \theta, \psi]\ \text{Coth}[\eta] + \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]}\ \text{Csc}[\theta]\ \text{Csch}[\eta]\ a\eta^{(0,0,1)}[\eta, \theta, \psi] - \right.$$
$$\left. \sqrt{2}\ a\psi^{(1,0,0)}[\eta, \theta, \psi] \right) \big/ \left(a\ \sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]} \right),$$
$$-\left(\left(\sqrt{2}\ \big(a\eta[\eta, \theta, \psi]\ \text{Sin}[2\,\theta] - a\theta[\eta, \theta, \psi]\ \text{Sinh}[2\,\eta] \big) - (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta]) \right) \right.$$
$$\left. \left. \big(a\eta^{(0,1,0)}[\eta, \theta, \psi] - a\theta^{(1,0,0)}[\eta, \theta, \psi] \big) \right) \big/ \left(a\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])^{3/2} \right) \right\}$$

Div[va, {η, θ, ψ}, {{"ProlateSpheroidal", {a}}, "Euclidean", 3}] // Expand //
 Simplify

$$-\left(\left(\sqrt{2}\ a\theta[\eta, \theta, \psi]\ (-1 + 2\,\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ \text{Cot}[\theta] + \right. \right.$$
$$\sqrt{2}\ a\eta[\eta, \theta, \psi]\ (1 + \text{Cos}[2\,\theta] - 2\,\text{Cosh}[2\,\eta])\ \text{Coth}[\eta] +$$
$$(\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ \left(\sqrt{-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta]}\ \text{Csc}[\theta]\ \text{Csch}[\eta]\ a\psi^{(0,0,1)}[\eta, \theta, \psi] + \right.$$
$$\left. \left. \left. \sqrt{2}\ \big(a\theta^{(0,1,0)}[\eta, \theta, \psi] + a\eta^{(1,0,0)}[\eta, \theta, \psi] \big) \right) \right) \big/ \left(a\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])^{3/2} \right) \right)$$

Laplacian[va, {η, θ, ψ}, {{"ProlateSpheroidal", {a}}, "Euclidean", 3}] // Expand

$$\left\{ -\frac{4\ a\eta[\eta, \theta, \psi]\ \text{Cos}[2\,\theta]}{a^2\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])^2} - \frac{4\ a\eta[\eta, \theta, \psi]\ \text{Cosh}[2\,\eta]}{a^2\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])^2} - \right.$$

$$\frac{4\ a\eta[\eta, \theta, \psi]\ \text{Cos}[2\,\theta]}{a^2\ (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])} -$$

$$\frac{4\ a\eta[\eta, \theta, \psi]\ \text{Cosh}[2\,\eta]}{a^2\ (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])} -$$

$$\frac{2\ a\eta[\eta, \theta, \psi]\ \text{Cot}[\theta]\ \text{Sin}[2\,\theta]}{a^2\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])^2} - \frac{2\ a\eta[\eta, \theta, \psi]\ \text{Cot}[\theta]\ \text{Sin}[2\,\theta]}{a^2\ (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])} -$$

$$\frac{2\ a\theta[\eta, \theta, \psi]\ \text{Coth}[\eta]\ \text{Sin}[2\,\theta]}{a^2\ (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])} + \frac{10\ a\eta[\eta, \theta, \psi]\ \text{Sin}[2\,\theta]^2}{a^2\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])^3} +$$

$$\frac{12\ a\eta[\eta, \theta, \psi]\ \text{Sin}[2\,\theta]^2}{a^2\ (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])^2} +$$

$$\frac{2\ a\theta[\eta, \theta, \psi]\ \text{Cot}[\theta]\ \text{Sinh}[2\,\eta]}{a^2\ (\text{Cos}[2\,\theta] - \text{Cosh}[2\,\eta])\ (-\text{Cos}[2\,\theta] + \text{Cosh}[2\,\eta])} -$$

$$\frac{2\,a\eta[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Sinh}[2\,\eta]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-\frac{2\,a\eta[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Sinh}[2\,\eta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cot}[\theta]\,\text{Csch}[\eta]^2\,\text{Sinh}[2\,\eta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])}+\frac{a\eta[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Csch}[\eta]^2\,\text{Sinh}[2\,\eta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])}+$$

$$\frac{10\,a\eta[\eta,\theta,\psi]\,\text{Sinh}[2\,\eta]^2}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^3}+\frac{12\,a\eta[\eta,\theta,\psi]\,\text{Sinh}[2\,\eta]^2}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}+$$

$$\left(\sqrt{2}\,\sqrt{-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta]}\,\text{Csc}[\theta]\,\text{Csch}[\eta]^3\,\text{Sinh}[2\,\eta]\,a\psi^{(0,0,1)}[\eta,\theta,\psi]\right)\Big/$$

$$\left(a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\right)+\frac{\text{Csc}[\theta]^2\,\text{Csch}[\eta]^2\,a\eta^{(0,0,2)}[\eta,\theta,\psi]}{a^2}+$$

$$\frac{2\,\text{Cot}[\theta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}-\frac{8\,\text{Sin}[2\,\theta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-$$

$$\frac{8\,\text{Sin}[2\,\theta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+$$

$$\frac{4\,\text{Sinh}[2\,\eta]\,a\theta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+\frac{2\,a\eta^{(0,2,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+$$

$$\frac{2\,\text{Coth}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}-\frac{8\,\text{Sinh}[2\,\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-$$

$$\frac{8\,\text{Sinh}[2\,\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}-$$

$$\frac{4\,\text{Sin}[2\,\theta]\,a\theta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+\frac{2\,a\eta^{(2,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}\,,$$

$$-\frac{4\,a\theta[\eta,\theta,\psi]\,\text{Cos}[2\,\theta]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-\frac{4\,a\theta[\eta,\theta,\psi]\,\text{Cosh}[2\,\eta]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-$$

$$\frac{4\,a\theta[\eta,\theta,\psi]\,\text{Cos}[2\,\theta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}-$$

$$\frac{4\,a\theta[\eta,\theta,\psi]\,\text{Cosh}[2\,\eta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}-$$

$$\frac{2\,a\theta[\eta,\theta,\psi]\,\text{Cot}[\theta]\,\text{Sin}[2\,\theta]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-\frac{2\,a\theta[\eta,\theta,\psi]\,\text{Cot}[\theta]\,\text{Sin}[2\,\theta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+$$

$$\frac{2\,a\eta[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Sin}[2\,\theta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2\,\text{Sin}[2\,\theta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])}+\frac{a\eta[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Csc}[\theta]^2\,\text{Sin}[2\,\theta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])}+$$

$$\frac{10\,a\theta[\eta,\theta,\psi]\,\text{Sin}[2\,\theta]^2}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^3}+\frac{12\,a\theta[\eta,\theta,\psi]\,\text{Sin}[2\,\theta]^2}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-$$

$$\frac{2\,a\eta[\eta,\theta,\psi]\,\text{Cot}[\theta]\,\text{Sinh}[2\,\eta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}-\frac{2\,a\theta[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Sinh}[2\,\eta]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}-$$

$$\frac{2\,a\theta[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Sinh}[2\,\eta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+\frac{10\,a\theta[\eta,\theta,\psi]\,\text{Sinh}[2\,\eta]^2}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^3}+$$

$$\frac{12\,a\theta[\eta,\theta,\psi]\,\text{Sinh}[2\,\eta]^2}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2}+$$

$$\left(\sqrt{2}\,\sqrt{-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta]}\,\text{Csc}[\theta]^3\,\text{Csch}[\eta]\,\text{Sin}[2\,\theta]\,a\psi^{(0,0,1)}[\eta,\theta,\psi]\right)\Big/$$

$$\left(a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\right)+\frac{\text{Csc}[\theta]^2\,\text{Csch}[\eta]^2\,a\theta^{(0,0,2)}[\eta,\theta,\psi]}{a^2}-$$

$$\frac{4\,\text{Sinh}[2\,\eta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}+\frac{2\,\text{Cot}[\theta]\,a\theta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}-$$

$$\frac{8\,\text{Sin}[2\,\theta]\,a\theta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2} - \frac{8\,\text{Sin}[2\,\theta]\,a\theta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} +$$

$$\frac{2\,a\theta^{(0,2,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} + \frac{4\,\text{Sin}[2\,\theta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} +$$

$$\frac{2\,\text{Coth}[\eta]\,a\theta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} - \frac{8\,\text{Sinh}[2\,\eta]\,a\theta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2} -$$

$$\frac{8\,\text{Sinh}[2\,\eta]\,a\theta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} + \frac{2\,a\theta^{(2,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])},$$

$$\frac{a\psi[\eta,\theta,\psi]\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2\,\text{Sin}[2\,\theta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])} + \frac{a\psi[\eta,\theta,\psi]\,\text{Coth}[\eta]\,\text{Csch}[\eta]^2\,\text{Sinh}[2\,\eta]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])} +$$

$$\frac{2\,\sqrt{2}\,\text{Coth}[\eta]\,\text{Csc}[\theta]\,\text{Csch}[\eta]\,a\eta^{(0,0,1)}[\eta,\theta,\psi]}{a^2\,\sqrt{-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta]}} +$$

$$\frac{2\,\sqrt{2}\,\text{Cot}[\theta]\,\text{Csc}[\theta]\,\text{Csch}[\eta]\,a\theta^{(0,0,1)}[\eta,\theta,\psi]}{a^2\,\sqrt{-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta]}} + \frac{\text{Csc}[\theta]^2\,\text{Csch}[\eta]^2\,a\psi^{(0,0,2)}[\eta,\theta,\psi]}{a^2} +$$

$$\frac{2\,\text{Cot}[\theta]\,a\psi^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} - \frac{4\,\text{Sin}[2\,\theta]\,a\psi^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2} -$$

$$\frac{4\,\text{Sin}[2\,\theta]\,a\psi^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} + \frac{2\,a\psi^{(0,2,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} +$$

$$\frac{2\,\text{Coth}[\eta]\,a\psi^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} - \frac{4\,\text{Sinh}[2\,\eta]\,a\psi^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])^2} -$$

$$\frac{4\,\text{Sinh}[2\,\eta]\,a\psi^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[2\,\theta]-\text{Cosh}[2\,\eta])\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])} + \frac{2\,a\psi^{(2,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[2\,\theta]+\text{Cosh}[2\,\eta])}\Big\}$$

```
{xp, yp, zp} = CoordinateTransformData[
  {{"ProlateSpheroidal", {a}}, "Euclidean", 3} → "Cartesian", "Mapping", {η, θ, ψ}]
```

$\{a\,\text{Cos}[\psi]\,\text{Sin}[\theta]\,\text{Sinh}[\eta],\ a\,\text{Sin}[\theta]\,\text{Sin}[\psi]\,\text{Sinh}[\eta],\ a\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\}$

```
{ηp, θp, ψp} = CoordinateTransformData["Cartesian" →
  {{"ProlateSpheroidal", {a}}, "Euclidean", 3}, "Mapping", {x, y, z}]
```

$$\Big\{\text{ArcCosh}\Big[\frac{\sqrt{a^2+x^2+y^2+z^2+\sqrt{x^2+y^2+(-a+z)^2}\,\sqrt{x^2+y^2+(a+z)^2}}}{\sqrt{2}\,a}\Big],$$

$$\frac{\pi}{2}-\text{ArcTan}\Big[\sqrt{2}\,z\,\sqrt{\Big(1\Big/\Big(a^2+x^2+y^2-z^2+\sqrt{(x^2+y^2+(a-z)^2)\,(x^2+y^2+(a+z)^2)}\Big)\Big)}\Big]\Big],$$

$$\text{ArcTan}[x, y]\Big\}$$

```
{rp, θp, ψp} = CoordinateTransformData[
  {{"ProlateSpheroidal", {a}}, "Euclidean", 3} → "Spherical", "Mapping", {η, θ, ψ}]
```

$$\Big\{\sqrt{\big(a^2\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2+a^2\,\text{Cos}[\psi]^2\,\text{Sin}[\theta]^2\,\text{Sinh}[\eta]^2+a^2\,\text{Sin}[\theta]^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2\big)},$$

$$\text{ArcTan}\big[a\,\text{Cos}[\theta]\,\text{Cosh}[\eta],$$

$$\sqrt{\big(a^2\,\text{Cos}[\psi]^2\,\text{Sin}[\theta]^2\,\text{Sinh}[\eta]^2+a^2\,\text{Sin}[\theta]^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2\big)}\big],$$

$$\text{ArcTan}[a\,\text{Cos}[\psi]\,\text{Sin}[\theta]\,\text{Sinh}[\eta],\ a\,\text{Sin}[\theta]\,\text{Sin}[\psi]\,\text{Sinh}[\eta]]\Big\}$$

```
{rp, Θp, ψp} =
 CoordinateTransformData["Cartesian" → "Spherical", "Mapping", {xp, yp, zp}]
```

$\left\{\sqrt{\left(a^2 \text{Cos}[\Theta]^2 \text{Cosh}[\eta]^2 + a^2 \text{Cos}[\psi]^2 \text{Sin}[\Theta]^2 \text{Sinh}[\eta]^2 + a^2 \text{Sin}[\Theta]^2 \text{Sin}[\psi]^2 \text{Sinh}[\eta]^2\right)},\right.$
$\text{ArcTan}\left[a \text{Cos}[\Theta] \text{Cosh}[\eta],\right.$
$\left.\sqrt{\left(a^2 \text{Cos}[\psi]^2 \text{Sin}[\Theta]^2 \text{Sinh}[\eta]^2 + a^2 \text{Sin}[\Theta]^2 \text{Sin}[\psi]^2 \text{Sinh}[\eta]^2\right)}\right],$
$\left.\text{ArcTan}[a \text{Cos}[\psi] \text{Sin}[\Theta] \text{Sinh}[\eta], a \text{Sin}[\Theta] \text{Sin}[\psi] \text{Sinh}[\eta]]\right\}$

B3.15 Toroidal

The coordinate surfaces are toroids (η = const.), spherical bowls (θ = const.) and vertical planes
(ψ = const.) passing through the z-axis. The radius of the focal circle $\sqrt{x^2 + y^2}$ is denoted as
a in accordance with Fig.4.04 of ref.[Moon, Spencer]. The coordinate η is a "quasi-radius"; θ is
called the poloidal coordinate, it gives an angle around the focal circle. ψ is the azimuthal angle,
sometimes called the toroidal coordinate.

```
{xp, yp, zp} = CoordinateTransformData[
  {{"Toroidal", {a}}, "Euclidean", 3} → "Cartesian", "Mapping", {η, θ, ψ}]
```

$\left\{\dfrac{a \text{Cos}[\psi] \text{Sinh}[\eta]}{-\text{Cos}[\Theta] + \text{Cosh}[\eta]}, \dfrac{a \text{Sin}[\psi] \text{Sinh}[\eta]}{-\text{Cos}[\Theta] + \text{Cosh}[\eta]}, \dfrac{a \text{Sin}[\Theta]}{-\text{Cos}[\Theta] + \text{Cosh}[\eta]}\right\}$

```
CoordinateChartData[
 {{"Toroidal", {a}}, "Euclidean", 3}, "ScaleFactors", {η, θ, ψ}]
```

$\left\{\dfrac{a}{-\text{Cos}[\Theta] + \text{Cosh}[\eta]}, \dfrac{a}{-\text{Cos}[\Theta] + \text{Cosh}[\eta]}, \dfrac{a \text{Sinh}[\eta]}{-\text{Cos}[\Theta] + \text{Cosh}[\eta]}\right\}$

```
Grad[Ψ[η, θ, ψ], {η, θ, ψ}, {{"Toroidal", {a}}, "Euclidean", 3}]
```

$\left\{\dfrac{(-\text{Cos}[\Theta] + \text{Cosh}[\eta]) \Psi^{(1,0,0)}[\eta, \Theta, \psi]}{a}, \dfrac{(-\text{Cos}[\Theta] + \text{Cosh}[\eta]) \Psi^{(0,1,0)}[\eta, \Theta, \psi]}{a},\right.$
$\left.\dfrac{(-\text{Cos}[\Theta] + \text{Cosh}[\eta]) \text{Csch}[\eta] \Psi^{(0,0,1)}[\eta, \Theta, \psi]}{a}\right\}$

```
Laplacian[Ψ[η, θ, ψ], {η, θ, ψ}, {{"Toroidal", {a}}, "Euclidean", 3}] // Expand //
 Simplify
```

$\dfrac{1}{a^2} (\text{Cos}[\Theta] - \text{Cosh}[\eta])$

$\left((\text{Cos}[\Theta] - \text{Cosh}[\eta]) \text{Csch}[\eta]^2 \Psi^{(0,0,2)}[\eta, \Theta, \psi] + \text{Sin}[\Theta] \Psi^{(0,1,0)}[\eta, \Theta, \psi] + \right.$
$\quad \text{Cos}[\Theta] \Psi^{(0,2,0)}[\eta, \Theta, \psi] - \text{Cosh}[\eta] \Psi^{(0,2,0)}[\eta, \Theta, \psi] + \text{Cos}[\Theta] \text{Coth}[\eta] \Psi^{(1,0,0)}[\eta, \Theta, \psi] -$
$\quad \text{Csch}[\eta] \Psi^{(1,0,0)}[\eta, \Theta, \psi] + \text{Cos}[\Theta] \Psi^{(2,0,0)}[\eta, \Theta, \psi] - \text{Cosh}[\eta] \Psi^{(2,0,0)}[\eta, \Theta, \psi]\left.\right)$

```
va = {aη[η, θ, ψ], aθ[η, θ, ψ], aψ[η, θ, ψ]}
```

$\{a\eta[\eta, \Theta, \psi], a\Theta[\eta, \Theta, \psi], a\psi[\eta, \Theta, \psi]\}$

Curl[va, {η, θ, ψ}, {{"Toroidal", {a}}, "Euclidean", 3}] // Expand // Simplify

$\{ \frac{1}{a} \big(-a\psi[\eta, \theta, \psi] \operatorname{Sin}[\theta] +$

$\quad (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) \big(\operatorname{Csch}[\eta] a\theta^{(0,0,1)}[\eta, \theta, \psi] - a\psi^{(0,1,0)}[\eta, \theta, \psi] \big) \big),$

$\quad \frac{1}{a} \big(a\psi[\eta, \theta, \psi] \big((\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta]) \operatorname{Coth}[\eta] - 2 \operatorname{Csch}[\eta] - \operatorname{Sinh}[\eta] \big) +$

$\quad (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) \operatorname{Csch}[\eta] \big(-a\eta^{(0,0,1)}[\eta, \theta, \psi] + \operatorname{Sinh}[\eta] a\psi^{(1,0,0)}[\eta, \theta, \psi] \big) \big),$

$\quad \frac{1}{a} \big(a\eta[\eta, \theta, \psi] \operatorname{Sin}[\theta] - a\theta[\eta, \theta, \psi] \operatorname{Sinh}[\eta] +$

$\quad (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) \big(a\eta^{(0,1,0)}[\eta, \theta, \psi] - a\theta^{(1,0,0)}[\eta, \theta, \psi] \big) \big) \}$

Div[va, {η, θ, ψ}, {{"Toroidal", {a}}, "Euclidean", 3}] // Expand // Simplify

$\frac{1}{2a} \big(a\eta[\eta, \theta, \psi] \big(-(2 \operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta]) \operatorname{Coth}[\eta] + 3 \operatorname{Csch}[\eta] - \operatorname{Sinh}[\eta] \big) -$

$\quad 2 \big(2 a\theta[\eta, \theta, \psi] \operatorname{Sin}[\theta] + (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])$

$\quad\quad \big(\operatorname{Csch}[\eta] a\psi^{(0,0,1)}[\eta, \theta, \psi] + a\theta^{(0,1,0)}[\eta, \theta, \psi] + a\eta^{(1,0,0)}[\eta, \theta, \psi] \big) \big) \big)$

Laplacian[va, {η, θ, ψ}, {{"Toroidal", {a}}, "Euclidean", 3}] // Expand

$\{ - \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} + \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^3}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])} + \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Cosh}[\eta]}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} +$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^3 \operatorname{Cosh}[\eta]}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} - \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2 \operatorname{Cosh}[\eta]}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])} -$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cosh}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} - \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2 \operatorname{Cosh}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} -$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Cosh}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])} + \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Cosh}[\eta]^3}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} + \frac{a\eta[\eta, \theta, \psi] \operatorname{Cosh}[\eta]^3}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])} +$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^3}{a^2 (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} - \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} -$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2 \operatorname{Cosh}[\eta]}{a^2 (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} + \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Cosh}[\eta]}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} +$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^3 \operatorname{Cosh}[\eta]}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} -$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Cosh}[\eta]^2}{a^2 (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} - \frac{a\eta[\eta, \theta, \psi] \operatorname{Cosh}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} -$

$\quad \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2 \operatorname{Cosh}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} + \frac{a\eta[\eta, \theta, \psi] \operatorname{Cosh}[\eta]^3}{a^2 (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} +$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Cosh}[\eta]^3}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta]) (-\operatorname{Cos}[\theta] + \operatorname{Cosh}[\eta])} - \frac{a\eta[\eta, \theta, \psi] \operatorname{Coth}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} -$

$\quad \frac{4 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2 \operatorname{Coth}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} - \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^4 \operatorname{Coth}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} +$

$\quad \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Cosh}[\eta] \operatorname{Coth}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} + \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^3 \operatorname{Cosh}[\eta] \operatorname{Coth}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} -$

$\quad \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2 \operatorname{Cosh}[\eta]^2 \operatorname{Coth}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} + \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta] \operatorname{Coth}[\eta] \operatorname{Csch}[\eta]}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} +$

$\quad \frac{2 a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^3 \operatorname{Coth}[\eta] \operatorname{Csch}[\eta]}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} - \frac{a\eta[\eta, \theta, \psi] \operatorname{Cos}[\theta]^2 \operatorname{Csch}[\eta]^2}{a^2 (\operatorname{Cos}[\theta] - \operatorname{Cosh}[\eta])^2} +$

$$\frac{4\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} - \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{8\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{4\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} -$$

$$\frac{2\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} + \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} -$$

$$\frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{6\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{12\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{6\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{a\theta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,\text{Sinh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} +$$

$$\frac{2\,a\theta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{Sinh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} - \frac{a\theta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,\text{Sinh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} +$$

$$\frac{3\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} - \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{6\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{3\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} -$$

$$\frac{2\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} + \frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} -$$

$$\frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{5\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{10\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{5\,a\eta[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{2\,\text{Coth}[\eta]^2\,a\psi^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{4\,\text{Cos}[\theta]^2\,\text{Coth}[\eta]^2\,a\psi^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2\,a\psi^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{4\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{Csch}[\eta]\,a\psi^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{2\,\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,\text{Csch}[\eta]\,a\psi^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Csch}[\eta]^2\,a\psi^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{\text{Coth}[\eta]^2\,a\eta^{(0,0,2)}[\eta,\,\theta,\,\psi]}{a^2} -$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{Csch}[\eta]\,a\eta^{(0,0,2)}[\eta,\,\theta,\,\psi]}{a^2} + \frac{\text{Cos}[\theta]^2\,\text{Csch}[\eta]^2\,a\eta^{(0,0,2)}[\eta,\,\theta,\,\psi]}{a^2} -$$

$$\frac{\text{Cos}[\theta]\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\,\theta,\,\psi]}{a^2} + \frac{5\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{\text{Cosh}[\eta]\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\,\theta,\,\psi]}{a^2} - \frac{10\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{5\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{3\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{6\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{3\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,a\eta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,a\theta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{4\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,a\theta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{2\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,a\theta^{(0,1,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{\text{Cos}[\theta]^2\,a\eta^{(0,2,0)}[\eta,\theta,\psi]}{a^2} -$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,a\eta^{(0,2,0)}[\eta,\theta,\psi]}{a^2} + \frac{\text{Cosh}[\eta]^2\,a\eta^{(0,2,0)}[\eta,\theta,\psi]}{a^2} +$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{\text{Cosh}[\eta]\,\text{Coth}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{2\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]\,\text{Coth}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{\text{Cos}[\theta]\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{\text{Cos}[\theta]^2\,\text{Csch}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{\text{Cos}[\theta]\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2} + \frac{4\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2} - \frac{8\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{4\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{3\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{6\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{3\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,a\eta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,a\theta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{4\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,a\theta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{2\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,a\theta^{(1,0,0)}[\eta,\theta,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{\text{Cos}[\theta]^2\,a\eta^{(2,0,0)}[\eta,\theta,\psi]}{a^2} -$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,a\eta^{(2,0,0)}[\eta,\theta,\psi]}{a^2} + \frac{\text{Cosh}[\eta]^2\,a\eta^{(2,0,0)}[\eta,\theta,\psi]}{a^2},$$

$$-\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^3}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{2\,a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} +$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} - \frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} - \frac{2\,a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} -$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^3}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\theta[\eta,\theta,\psi]\,\text{Cosh}[\eta]^3}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^3}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{2\,a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{a\theta[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{a\theta[\eta,\theta,\psi]\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{2\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Cosh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}+\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^3}{a^2\,(-Cos[\Theta]+Cosh[\eta])}+$$

$$\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]^3}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}-\frac{4\,a\eta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Coth[\eta]\,Sin[\Theta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}-$$

$$\frac{2\,a\eta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^3\,Coth[\eta]\,Sin[\Theta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+\frac{2\,a\eta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]\,Coth[\eta]\,Sin[\Theta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+$$

$$\frac{4\,a\eta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Cosh[\eta]\,Coth[\eta]\,Sin[\Theta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}-$$

$$\frac{2\,a\eta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]^2\,Coth[\eta]\,Sin[\Theta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+\frac{2\,a\eta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Csch[\eta]\,Sin[\Theta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+$$

$$\frac{3\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}-\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])}-$$

$$\frac{6\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])}+$$

$$\frac{3\,a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^2\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Sin[\Theta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])^2}-$$

$$\frac{2\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]\,Sin[\Theta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])^2}+\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^2\,Sin[\Theta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])^2}-$$

$$\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Sin[\Theta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])}+\frac{6\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}+$$

$$\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]\,Sin[\Theta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])}-\frac{12\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}+$$

$$\frac{6\,a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^2\,Sin[\Theta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}+\frac{a\eta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Sin[\Theta]\,Sinh[\eta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}-$$

$$\frac{2\,a\eta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]\,Sin[\Theta]\,Sinh[\eta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+\frac{a\eta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^2\,Sin[\Theta]\,Sinh[\eta]}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+$$

$$\frac{3\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}-\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])}-$$

$$\frac{6\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])}+$$

$$\frac{3\,a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^2\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])^2}+\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Sinh[\eta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])^2}-$$

$$\frac{2\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]\,Sinh[\eta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])^2}+\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^2\,Sinh[\eta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])^2}-$$

$$\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Sinh[\eta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])}+\frac{5\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]^2\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}+$$

$$\frac{a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]\,Sinh[\eta]^2}{a^2\,(-Cos[\Theta]+Cosh[\eta])}-\frac{10\,a\theta[\eta,\,\Theta,\,\psi]\,Cos[\Theta]\,Cosh[\eta]\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}+$$

$$\frac{5\,a\theta[\eta,\,\Theta,\,\psi]\,Cosh[\eta]^2\,Sinh[\eta]^2}{a^2\,(Cos[\Theta]-Cosh[\eta])\,(-Cos[\Theta]+Cosh[\eta])}+$$

$$\frac{4\,Cos[\Theta]\,Coth[\eta]\,Sin[\Theta]\,a\psi^{(0,0,1)}[\eta,\,\Theta,\,\psi]}{a^2\,(Cos[\Theta]-Cosh[\eta])}-$$

$$\frac{2\,Cosh[\eta]\,Coth[\eta]\,Sin[\Theta]\,a\psi^{(0,0,1)}[\eta,\,\Theta,\,\psi]}{a^2\,(Cos[\Theta]-Cosh[\eta])}-$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Csch}[\eta]\,\text{Sin}[\theta]\,\text{a}\psi^{(0,0,1)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{\text{Coth}[\eta]^2\,\text{a}\theta^{(0,0,2)}[\eta,\theta,\psi]}{\text{a}^2} -$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{Csch}[\eta]\,\text{a}\theta^{(0,0,2)}[\eta,\theta,\psi]}{\text{a}^2} + \frac{\text{Cos}[\theta]^2\,\text{Csch}[\eta]^2\,\text{a}\theta^{(0,0,2)}[\eta,\theta,\psi]}{\text{a}^2} +$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,\text{a}\eta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{4\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,\text{a}\eta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{2\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,\text{a}\eta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{\text{Cos}[\theta]\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2} +$$

$$\frac{5\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2} -$$

$$\frac{10\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{5\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{3\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{6\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{3\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,\text{a}\theta^{(0,1,0)}[\eta,\theta,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{\text{Cos}[\theta]^2\,\text{a}\theta^{(0,2,0)}[\eta,\theta,\psi]}{\text{a}^2} -$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{a}\theta^{(0,2,0)}[\eta,\theta,\psi]}{\text{a}^2} + \frac{\text{Cosh}[\eta]^2\,\text{a}\theta^{(0,2,0)}[\eta,\theta,\psi]}{\text{a}^2} -$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,\text{a}\eta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{4\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{a}\eta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{2\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,\text{a}\eta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{2\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{\text{Cosh}[\eta]\,\text{Coth}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]\,\text{Coth}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{\text{Cos}[\theta]\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{\text{Cos}[\theta]^2\,\text{Csch}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{\text{Cos}[\theta]\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2} + \frac{4\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2} - \frac{8\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{4\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{3\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{6\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{3\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,\text{a}\theta^{(1,0,0)}[\eta,\theta,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{\text{Cos}[\theta]^2\,\text{a}\theta^{(2,0,0)}[\eta,\theta,\psi]}{\text{a}^2} -$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{a}\theta^{(2,0,0)}[\eta,\theta,\psi]}{\text{a}^2} + \frac{\text{Cosh}[\eta]^2\,\text{a}\theta^{(2,0,0)}[\eta,\theta,\psi]}{\text{a}^2},$$

$$-\frac{3\,\text{a}\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{\text{a}\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{\text{a}\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{6\,\text{a}\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{3\,\text{a}\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} -$$

$$\frac{\text{a}\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{3\,\text{a}\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} -$$

$$\frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} - \frac{6\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} +$$

$$\frac{2\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} + \frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^3}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} +$$

$$\frac{2\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} - \frac{6\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} +$$

$$\frac{6\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^3}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} - \frac{2\,a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]^4}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} +$$

$$\frac{a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{4\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{8\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{6\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{4\,a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{14\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]^3}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{10\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^3}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{2\,a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]^4}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{a\psi[\eta,\theta,\psi]\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} +$$

$$\frac{4\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^4\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} -$$

$$\frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} - \frac{2\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} -$$

$$\frac{2\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])} +$$

$$\frac{a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2} + \frac{a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^4\,\text{Coth}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} -$$

$$\frac{4\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} + \frac{6\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} -$$

$$\frac{4\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^3\,\text{Coth}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} + \frac{a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]^4\,\text{Coth}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2} -$$

$$\frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Coth}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} + \frac{9\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^2\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^4\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} - \frac{9\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} -$$

$$\frac{9\,a\psi[\eta,\theta,\psi]\,\text{Cos}[\theta]^3\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{3\,a\psi[\eta,\theta,\psi]\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])} +$$

$$\frac{9\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-$$

$$\frac{3\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^3\,\text{Coth}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-$$

$$\frac{2\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{Csch}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2}-\frac{2\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,\text{Csch}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2}+$$

$$\frac{3\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Coth}[\eta]\,\text{Csch}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+\frac{3\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Coth}[\eta]\,\text{Csch}[\eta]}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-$$

$$\frac{3\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,\text{Csch}[\eta]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Csch}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2}-$$

$$\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^3\,\text{Csch}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^3\,\text{Csch}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{4\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2}-\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-$$

$$\frac{8\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2}+\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{4\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])^2}+\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2}-$$

$$\frac{2\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2}+\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2}-$$

$$\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+\frac{6\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-\frac{12\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{6\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2}-$$

$$\frac{2\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2}+\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])^2}-$$

$$\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+\frac{2\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-\frac{4\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{2\,a\psi[\eta,\,\theta,\,\psi]\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]^2}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-\frac{2\,\text{Coth}[\eta]^2\,a\eta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-$$

$$\frac{4\,\text{Cos}[\theta]^2\,\text{Coth}[\eta]^2\,a\eta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+\frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Coth}[\eta]^2\,a\eta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{4\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{Csch}[\eta]\,a\eta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{2\,\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,\text{Csch}[\eta]\,a\eta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Csch}[\eta]^2\,a\eta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-\frac{4\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{Sin}[\theta]\,a\theta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{2\,\text{Cosh}[\eta]\,\text{Coth}[\eta]\,\text{Sin}[\theta]\,a\theta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{a^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Csch}[\eta]\,\text{Sin}[\theta]\,\text{a}\theta^{(0,0,1)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+\frac{\text{Coth}[\eta]^2\,\text{a}\psi^{(0,0,2)}[\eta,\,\theta,\,\psi]}{\text{a}^2}-$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{Csch}[\eta]\,\text{a}\psi^{(0,0,2)}[\eta,\,\theta,\,\psi]}{\text{a}^2}+\frac{\text{Cos}[\theta]^2\,\text{Csch}[\eta]^2\,\text{a}\psi^{(0,0,2)}[\eta,\,\theta,\,\psi]}{\text{a}^2}-$$

$$\frac{\text{Cos}[\theta]\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}+\frac{5\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}-\frac{10\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{5\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+\frac{3\,\text{Cos}[\theta]^2\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-$$

$$\frac{6\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+\frac{3\,\text{Cosh}[\eta]^2\,\text{Sin}[\theta]\,\text{a}\psi^{(0,1,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{\text{Cos}[\theta]^2\,\text{a}\psi^{(0,2,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}-\frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{a}\psi^{(0,2,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}+$$

$$\frac{\text{Cosh}[\eta]^2\,\text{a}\psi^{(0,2,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}+\frac{6\,\text{Cos}[\theta]\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{3\,\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-\frac{3\,\text{Cosh}[\eta]\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-$$

$$\frac{6\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{3\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{2\,\text{Cos}[\theta]^3\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-\frac{6\,\text{Cos}[\theta]^2\,\text{Cosh}[\eta]\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{6\,\text{Cos}[\theta]\,\text{Cosh}[\eta]^2\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-\frac{2\,\text{Cosh}[\eta]^3\,\text{Coth}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-$$

$$\frac{3\,\text{Cos}[\theta]^2\,\text{Csch}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}-\frac{\text{Cos}[\theta]\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}+$$

$$\frac{2\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+\frac{\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}-$$

$$\frac{4\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+\frac{2\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(\text{Cos}[\theta]-\text{Cosh}[\eta])}+$$

$$\frac{3\,\text{Cos}[\theta]^2\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}-\frac{6\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+$$

$$\frac{3\,\text{Cosh}[\eta]^2\,\text{Sinh}[\eta]\,\text{a}\psi^{(1,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2\,(-\text{Cos}[\theta]+\text{Cosh}[\eta])}+\frac{\text{Cos}[\theta]^2\,\text{a}\psi^{(2,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}-$$

$$\frac{2\,\text{Cos}[\theta]\,\text{Cosh}[\eta]\,\text{a}\psi^{(2,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}+\frac{\text{Cosh}[\eta]^2\,\text{a}\psi^{(2,0,0)}[\eta,\,\theta,\,\psi]}{\text{a}^2}\Big\}$$

```
{xp, yp, zp} = CoordinateTransformData[
  {{"Toroidal", {a}}, "Euclidean", 3} → "Cartesian", "Mapping", {η, θ, ψ}]
```

$$\left\{\frac{\text{a}\,\text{Cos}[\psi]\,\text{Sinh}[\eta]}{-\text{Cos}[\theta]+\text{Cosh}[\eta]},\ \frac{\text{a}\,\text{Sin}[\psi]\,\text{Sinh}[\eta]}{-\text{Cos}[\theta]+\text{Cosh}[\eta]},\ \frac{\text{a}\,\text{Sin}[\theta]}{-\text{Cos}[\theta]+\text{Cosh}[\eta]}\right\}$$

```
{ηp, Θp, ψp} = CoordinateTransformData[
   "Cartesian" → {{"Toroidal", {a}}, "Euclidean", 3}, "Mapping", {x, y, z}]
```

$$\left\{ \frac{1}{2} \, \text{Log}\left[\frac{\left(a + \sqrt{x^2 + y^2}\right)^2 + z^2}{\left(-a + \sqrt{x^2 + y^2}\right)^2 + z^2} \right], \; \text{ArcTan}\left[-a^2 + x^2 + y^2 + z^2, \; 2\,a\,z\right], \; \text{ArcTan}[x, y] \right\}$$

```
{rp, Θp, ψp} = CoordinateTransformData[{{"Toroidal", {a}}, "Euclidean", 3} →
   {{"ProlateSpheroidal", {b}}, "Euclidean", 3}, "Mapping", {η, Θ, ψ}]
```

$$\Big\{ \text{ArcCosh}\Big[$$
$$\frac{1}{\sqrt{2}\,b}\left(\sqrt{\left(b^2 + \frac{a^2\,\text{Sin}[\Theta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \frac{a^2\,\text{Cos}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \frac{a^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \right.}\right.$$
$$\sqrt{\left(\left(-b + \frac{a\,\text{Sin}[\Theta]}{-\text{Cos}[\Theta]+\text{Cosh}[\eta]}\right)^2 + \frac{a^2\,\text{Cos}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \frac{a^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2}\right)}$$
$$\sqrt{\left(\left(b + \frac{a\,\text{Sin}[\Theta]}{-\text{Cos}[\Theta]+\text{Cosh}[\eta]}\right)^2 + \frac{a^2\,\text{Cos}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \right.}$$
$$\left.\left.\frac{a^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2}\right)\right)\Big], \; \frac{\pi}{2} - \text{ArcTan}\Big[\frac{1}{-\text{Cos}[\Theta]+\text{Cosh}[\eta]} \sqrt{2}\,a\,\text{Sin}[\Theta]$$
$$\sqrt{\left(1 \Big/ \left(b^2 - \frac{a^2\,\text{Sin}[\Theta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \frac{a^2\,\text{Cos}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \frac{a^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \right.\right.}$$
$$\sqrt{\left(\left(\left(b - \frac{a\,\text{Sin}[\Theta]}{-\text{Cos}[\Theta]+\text{Cosh}[\eta]}\right)^2 + \frac{a^2\,\text{Cos}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \right.\right.}$$
$$\left.\frac{a^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2}\right) \left(\left(b + \frac{a\,\text{Sin}[\Theta]}{-\text{Cos}[\Theta]+\text{Cosh}[\eta]}\right)^2 + \right.$$
$$\left.\left.\left.\left.\frac{a^2\,\text{Cos}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2} + \frac{a^2\,\text{Sin}[\psi]^2\,\text{Sinh}[\eta]^2}{(-\text{Cos}[\Theta]+\text{Cosh}[\eta])^2}\right)\right)\right)\right)\Big],$$
$$\text{ArcTan}\Big[\frac{a\,\text{Cos}[\psi]\,\text{Sinh}[\eta]}{-\text{Cos}[\Theta]+\text{Cosh}[\eta]}, \; \frac{a\,\text{Sin}[\psi]\,\text{Sinh}[\eta]}{-\text{Cos}[\Theta]+\text{Cosh}[\eta]} \Big]\Big\}$$

The R-separation of the Laplacian and the ensuing solutions are described in a separate file called VecAnalysisTor.pdf .

14.8 Tutorials, guides and notebooks for the Kernel Version of Vector analysis

ArcCurvature.nb
tutorial/ChangingCoordinateSystems
CoordinateChartData.nb
CoordinateTransform.nb
CoordinateTransformData.nb
Cross.nb
Curl.nb
Div.nb
Dot.nb
guide /FunctionsForSeparableCoordinateSystems
Laplacian.nb
guide/SpecialFunctions
TransformedField.nb
guide/VectorAnalysis