R-Separation in Toruskoordinaten

Toroidal coordinates η, θ, ϕ:

$$\begin{aligned} x &= \frac{a \sinh\eta \; \cos\phi}{\cosh\eta - \cos\theta} \\ y &= \frac{a \sinh\eta \; \sin\phi}{\cosh\eta - \cos\theta} \\ z &= \frac{a \sin\theta}{\cosh\eta - \cos\theta} \end{aligned} \tag{1}$$

The limit $\eta \to \infty$ gives the focal circle:

$$x^2 + y^2 \;=\; a^2. \tag{2}$$

The limit $\eta \to 0$ gives points on the z-axis:

$$(x, y, z) \;=\; (0, 0, \cot(\theta/2)), \quad z(\theta = \pi) = 0, \quad z(\theta = 0) = \infty; \tag{3}$$

in particular the origin for $\theta = \pi$, infinity for $\theta = 0$.

The geometric dimensions of a given torus are the major radius, or radius of curvature R_C = radius of the centre circle and the minor radius ρ_0. To these correspond unique values of the parameter a and η_0. These are given by:

$$R_c = a \; \cosh\eta_0, \quad \rho_0 = \frac{a}{\sinh\eta_0}; \quad \to \; \eta_0 = \text{Arcosh}(R_c/\rho_0). \tag{4}$$

The given torus $\eta = \eta_0$ divides the space into complimentary subspaces:

$$\text{Interior: } \infty > \eta \geq \eta_0; \qquad \text{Exterior: } \eta_0 \geq \eta \geq 0. \tag{5}$$

Potential equation in toroidal coordinates

The potential equation in toroidal coordinates η, θ, ϕ is:

$$\begin{aligned} a^2 \; \Delta V \;=\; & \frac{(\cosh\eta - \cos\theta)^3}{\sinh\eta} \left[\frac{\partial}{\partial\eta} \left(\frac{\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial}{\partial\eta} \right) \; + \right. \\ & \left. + \; \frac{\partial}{\partial\theta} \left(\frac{\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial}{\partial\theta} \right) \; + \; \frac{1/\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial^2}{\partial\phi^2} \right] V(\eta, \theta, \phi) = 0. \end{aligned} \tag{6}$$

R-separation

Attempt separation of variables by inserting the trial solution:

$$V(\eta, \theta, \phi) \;=\; \sqrt{\cosh\eta - \cos\theta} \; H(\eta) \; \Theta(\theta) \; \Phi(\phi), \tag{7}$$

then divide the result by $(\cosh\eta - \cos\theta)^{5/2}/\sinh^2\eta\ H(\eta)\ \Theta(\theta)\ \Phi(\phi)$ to obtain

$$\frac{1}{4}\sinh^2\eta + \cosh\eta\sinh\eta\frac{H'(\eta)}{H(\eta)} + \sinh^2\eta\,\frac{H''(\eta)}{H(\eta)} + \sinh^2\eta\,\frac{\Theta''(\eta)}{\Theta(\eta)} + \frac{\Phi''(\phi)}{\Phi(\phi)} \;=\; 0. \tag{8}$$

The function $\Phi(\phi)$ then separates with

$$\frac{\Phi''(\phi)}{\Phi(\phi)} \;=\; -\,m^2, \qquad m \in \mathbb{Z}. \tag{9}$$

Replacing $\frac{\Phi''(\phi)}{\Phi(\phi)}$ in eq.(8) by $-\,m^2$ and dividing the result by $\sinh^2\eta$ gives :

$$\coth\eta\,\frac{H'(\eta)}{H(\eta)} \;+\; \frac{H''(\eta)}{H(\eta)} \;-\; \frac{m^2}{\sinh^2\eta} \;+\; \frac{1}{4} \;+\; \frac{\Theta''(\eta)}{\Theta(\eta)} \;=\; 0. \tag{10}$$

The function $\Theta(\eta)$ then separates with

$$\frac{\Theta''(\eta)}{\Theta(\eta)} \;=\; -\,n^2, \qquad n \in \mathbb{Z}. \tag{11}$$

Replacing $\frac{\Theta''(\eta)}{\Theta(\eta)}$ in eq.(10) by $-\,n^2$ and multiplying the result by $\Theta(\eta)$ gives :

$$H''(\eta) \;+\; \coth\eta\;H'(\eta) \;-\; \left[\frac{m^2}{\sinh^2\eta} \;+\; \left(n^2 - \frac{1}{4}\right)\right] H(\eta) \;=\; 0. \tag{12}$$

Solutions

The solutions for $\Phi(\phi)$, $\Theta(\eta)$ respectively are trigonometric functions of argument $m\phi$, $n\theta$ respectively.

The solutions $H(\eta)$ can be expressed by Legendre functions. The Legendre differential equation for the functions $P^\mu_\nu(z)$ and $Q^\mu_\nu(z)$ is:

$$(z^2-1)\,\frac{d^2P^\mu_\nu(z)}{dz^2} \;+\; 2z\,\frac{dP^\mu_\nu(z)}{dz} \;-\; \left[\nu(\nu+1) \;+\; \frac{\mu^2}{z^2-1}\right]P^\mu_\nu(z) \;=\; 0. \tag{13}$$

Substituting $z = \cosh\eta$, applying the chain rule for the derivatives, and comparing the constants in the square brackets of eqs.(12) and (13) leads to:

$$\mu = m, \qquad \nu(\nu+1) = (n-1/2)(n+1/2) = n^2 - 1/4, \quad \nu = n - 1/2; \tag{14}$$

and

$$\begin{aligned}&\frac{d^2P^m_{n-1/2}(\cosh\eta)}{d\eta^2} \;+\; \coth\eta\,\frac{dP^m_{n-1/2}(\cosh\eta)}{d\eta} \;-\\ &-\;\left[\frac{m^2}{\sinh^2\eta} \;+\; \left(n^2 - \frac{1}{4}\right)\right] P^m_{n-1/2}(\cosh\eta) \;=\; 0,\end{aligned} \tag{15}$$

which agrees with eq.(12). All these equations are also valid for $Q^m_{n-1/2}(\cos h\eta)$.

Implementations in Mathematica

$$P^m_{n-1/2}(\cosh\eta) = LegendreP[n-1/2, m, 3, Cosh[\eta]], \qquad (16)$$
$$Q^m_{n-1/2}(\cosh\eta) = LegendreQ[n-1/2, m, 3, Cosh[\eta]]. \qquad (17)$$

$Q^m_{n-1/2}(\cosh\eta), n \geq 0, n \geq m \ \geq 0$ grows towards $\pm\infty$ as η tends to 0; it stays finite for finite positive η; so this function gives finite values in the interior of the torus. On the other hand, $P^m_{n-1/2}(\cosh\eta, n \geq 0, n \geq m \ \geq 0$ grows towards $\pm\infty$ if η tends to infinity; so this function gives infinite values in the interior, finite values in the exterior. An exception is $P_{-1/2}(\cosh\eta)$, which is finite for all $\eta \geq 1$.

References

[1] Weisstein, Eric W. : Laplace's Equation–Toroidal Coordinates.
From MathWorld–A Wolfram Web Resource.
http://mathworld.wolfram.com/LaplacesEquationToroidalCoordinates.html
[2] Moon, P., and Spencer, D.E.: Field Theory Handbook. Springer, 1971.
Fig.4.04, p.112 ff.