9.2.1 Beipiele zum Schmidtschen Orthogonalisierungsverfahren

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<<LinearAlgebra`Orthogonalization`
```

Legendrepolynome $P_n(x)$

■ Orthogonlisierung der Potentzen

Orthogonalisiert man die Potenzen von x im Intervall [-1,1], ergeben sich die Legendrepolynome $P_n(x)$.

```
lp = Table[x^k, {k, 0, 7}]
```

$$\{1, x, x^2, x^3, x^4, x^5, x^6, x^7\}$$

```
lgs = GramSchmidt[lp, InnerProduct → (Integrate[#1 #2, {x, -1, 1}] &)] // Simplify
```

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \frac{1}{2}\sqrt{\frac{5}{2}}\,(-1+3x^2), \frac{1}{2}\sqrt{\frac{7}{2}}\, x\,(-3+5x^2), \frac{3\,(3-30x^2+35x^4)}{8\sqrt{2}}, \frac{1}{8}\sqrt{\frac{11}{2}}\, x\,(15-70x^2+63x^4), \frac{1}{16}\sqrt{\frac{13}{2}}\,(-5+105x^2-315x^4+231x^6), \frac{1}{16}\sqrt{\frac{15}{2}}\, x\,(-35+315x^2-693x^4+429x^6)\right\}$$

```
Table[LegendreP[k,x],{k,0,Length[lp]-1}]//Simplify
```

$$\left\{1, x, \frac{1}{2}(-1+3x^2), \frac{1}{2}x(-3+5x^2), \frac{1}{8}(3-30x^2+35x^4), \frac{1}{8}x(15-70x^2+63x^4), \frac{1}{16}(-5+105x^2-315x^4+231x^6), \frac{1}{16}x(-35+315x^2-693x^4+429x^6)\right\}$$

```
lpp = Table[Normalize[LegendreP[k,x],
 InnerProduct ->
     (Integrate[#1 #2,{x,-1,1}]&)],{k,0,Length[lp]-1}]//Simplify;
lgs == lpp
```

True

■ Berechnung der Orthogonalität und Normierung mittels der erzeugenden Funktion

Die erzeugende Funktion der Legendre - Polynome ist :

$$\frac{1}{\sqrt{1-2x\alpha+\alpha^2}} = \sum_{n=0}^{\infty} \alpha^n P_n(x)$$

```
fa = 1/Sqrt[1 - 2 α x + α^2]
```

$$\frac{1}{\sqrt{1-2\,x\,\alpha+\alpha^2}}$$

```
fb = fa /. α → β
```

$$\frac{1}{\sqrt{1-2\,x\,\beta+\beta^2}}$$

```
in = Integrate[fa fb , {x, -1, 1}]
```

$$\text{If}\Big[\Big(\text{Re}\Big[\frac{1}{\alpha}+\alpha\Big]\le -2\ ||\ \text{Re}\Big[\frac{1}{\alpha}+\alpha\Big]\ge 2\ ||\ \text{Im}\Big[\frac{1}{\alpha}+\alpha\Big]\ne 0\Big)\ \&\&\ \Big(\text{Re}\Big[\frac{1}{\beta}+\beta\Big]\le -2\ ||\ \text{Re}\Big[\frac{1}{\beta}+\beta\Big]\ge 2\ ||\ \text{Im}\Big[\frac{1}{\beta}+\beta\Big]\ne 0\Big),$$

$$\frac{1}{2\sqrt{\alpha}\sqrt{\beta}}\left(\text{Log}\Big[\frac{2\sqrt{(-1+\alpha)^2}\sqrt{\alpha}\sqrt{(-1+\beta)^2}\sqrt{\beta}-\beta-\alpha^2\beta-\alpha\,(1-4\beta+\beta^2)}{\sqrt{\alpha}\sqrt{\beta}}\Big]-\text{Log}\Big[\frac{-\beta-\alpha^2\beta+2\sqrt{\alpha}\sqrt{(1+\alpha)^2}\sqrt{\beta}\sqrt{(1+\beta)^2}-\alpha\,(1+4\beta+\beta^2)}{\sqrt{\alpha}\sqrt{\beta}}\Big]\right),$$

$$\text{Integrate}\Big[\frac{1}{\sqrt{1-2\,x\,\alpha+\alpha^2}\sqrt{1-2\,x\,\beta+\beta^2}},\ \{x,-1,1\},\ \text{Assumptions}\to !\Big(\Big(\text{Re}\Big[\frac{1}{\alpha}+\alpha\Big]\le -2\ ||\ \text{Re}\Big[\frac{1}{\alpha}+\alpha\Big]\ge 2\ ||\ \text{Im}\Big[\frac{1}{\alpha}+\alpha\Big]\ne 0\Big)\ \&\&\ \Big(\text{Re}\Big[\frac{1}{\beta}+\beta\Big]\le -2\ ||\ \text{Re}\Big[\frac{1}{\beta}+\beta\Big]\ge 2\ ||\ \text{Im}\Big[\frac{1}{\beta}+\beta\Big]\ne 0\Big)\Big)\Big]\Big]$$

```
Simplify[in, 1 > α > 0 && 1 > β > 0]
```

$$\frac{1}{2\sqrt{\alpha\beta}}\left(-\text{Log}\Big[\frac{-\alpha-\beta-4\,\alpha\,\beta-\alpha^2\,\beta-\alpha\,\beta^2+2\,(1+\alpha)\sqrt{\alpha\,\beta}\,(1+\beta)}{\sqrt{\alpha\,\beta}}\Big]+\text{Log}\Big[\frac{-\beta-\alpha^2\,\beta+2\sqrt{(-1+\alpha)^2\,\alpha\,(-1+\beta)^2\,\beta}-\alpha\,(1-4\,\beta+\beta^2)}{\sqrt{\alpha\,\beta}}\Big]\right)$$

Händische Integration des obigen Integrals gibt:

$$\int_{-1}^{1}\frac{1}{\sqrt{1-2\,x\,\alpha+\alpha^2}}\ \frac{1}{\sqrt{1-2\,x\,\beta+\beta^2}}\,dx == \frac{1}{\sqrt{\alpha\beta}}\ \ln\frac{1+\sqrt{\alpha\beta}}{1-\sqrt{\alpha\beta}} =$$

$$=\sum_{n=0}^{\infty}\frac{2}{2\,n+1}\,(\alpha\,\beta)^n = \sum_{n=0}^{\infty}\alpha^n\sum_{k=0}^{\infty}\beta^k\int_{-1}^{1}dx\ P_n(x)\,P_k(x)$$