9.1.1.2 Anwendung der Shanks-Transformation zu Konvergenz-beschleunigung und -erzeugung.

The Shanks transformation is a non-linear sequence transformation. It can be used to accelerate the convergence of slowly converging series or to induce convergence of divergent series. It works best if the terms of the series change sign frequently, so in Fourier sine or cosine series in the middle of the interval. At the ends it performs poorly, it may even give inaccurate results for convergin series !

1. Shanks transformation, listings

```
shanks[ps_, n_] := Module[{aux}, e1 = ps; e2 = Table[0, {n}];
  Do[aux = Table[e2[[j + 1]] + 1 / (e1[[j + 1]] - e1[[j]]), {j, i}];
   e2 = e1; e1 = aux, {i, n - 1, 1, -1}]; If[Mod[n, 2] == 0, e2[[-1]], e1[[-1]]]]

shanksn[ps_, n_] := Module[{aux}, e1 = ps; e2 = Table[0, {n}];
      Do[aux = Table[Block[{diff = e1[[j + 1]] - e1[[j]]},
     If[Abs[diff] < 10^-15, e2[[j + 1]] + Sign[diff] 10^15, e2[[j + 1]] + 1 / diff]], {j, i}];
   e2 = e1; e1 = aux, {i, n - 1, 1, -1}]; If[Mod[n, 2] == 0, e2[[-1]], e1[[-1]]]]

General::spell1 : Possible spelling error: new
   symbol name "shanksn" is similar to existing symbol "shanks". More…
```

2. The example function and the Fourier coefficients

```
Clear[n, x, a0, fan, α, ff, fg];
fg[α_, x_] = (π^2 - x^2)^α / π^(2 α);
```

The Fourier coefficient a0:

$$a0[\alpha\_] = \frac{\pi^{\frac{1}{2}}\, \text{Gamma}[1+\alpha]}{\text{Gamma}\left[\frac{3}{2}+\alpha\right]};$$

The Fourier coefficients a_n , $n > 0$ are:

$$fan[\alpha\_, n\_] = 2^{\frac{1}{2}+\alpha}\, (n)^{\frac{1}{2}(-1-2\alpha)}\, \pi^{-\alpha}\, \text{BesselJ}\left[\frac{1}{2}+\alpha, n\,\pi\right] \text{Gamma}[1+\alpha];$$

The terms of the Fourier series:

```
rn[α_, n_, x_] := If[n == 0, a0[α] / 2, fan[α, n] Cos[n x] ]
```

- **2.1 Plots of the expansion coefficients (= Fourier coefficients for $\alpha > -0.5$)**

```
gfr = {Dashing[{.05, .02}], Line[{{-.5, 15}, {-.5, -15}}]};

pa0 = Plot[a0[α], {α, -1.5`, 1.5`},
  PlotRange → {{-1.5`, 1.5`}, 10 {-1, 1.5`}}, AxesLabel → {"α", a0}, Epilog → gfr]

pan = Table[Plot[fan[α, n], {α, -2.5, 5}, PlotRange → 5 {-1, 1}, Epilog → gfr,
     (* PlotStyle → Hue[0.1 n], *) AxesLabel → {"α", Subscript[a, n]}], {n, 9}];
```

$$\left(\pi^2 - x^2\right)^\alpha / \pi^{2\alpha} \sim$$

$$\frac{\pi^{\frac{1}{2}+2\alpha}\, \text{Gamma}[1+\alpha]}{\text{Gamma}\left[\frac{3}{2}+\alpha\right]} + \sum_{n=1}^{\infty} 2^{\frac{1}{2}+\alpha}\, \pi^{\alpha}\, \text{Gamma}[1+\alpha]\; n\_^{\frac{1}{2}(-1-2\alpha)}\; \text{BesselJ}\left[\frac{1}{2}+\alpha, \pi\, n\_\right]\; \text{Cos}[n\, x]$$

```
lip = Partition[Prepend[pan, pa0], 2];

Show[GraphicsGrid[lip], ImageSize → 500]
```

```
pa0 = Plot[a0[α], {α, -1.5`, 10}, AxesLabel → {"α", a0}, Epilog → gfr]

pan = Table[Plot[fan[α, n], {α, -1.5, 5}, PlotRange →  5 {-1, 1}, Epilog → gfr,
    (* PlotStyle → Hue[0.1 n], *) AxesLabel → {"α", Subscript[a, n]}], {n, 9}];

liq = Partition[Prepend[pan, pa0], 2];

Show[GraphicsGrid[liq], ImageSize → 500]
```

3. Point near to the middle of the interval

■ 3.1 Acceleration of convergence rate for a slowly converging Fourier series

```
nt = 30; (* number of terms *)
α = .5; x =  1.6;
ff[xx_] = fg[α, xx]; ff[x]
NIntegrate[ff[xx]^2, {xx, -π, π}]
```

```
0.860592

4.18879
```

■ **Terms of the series**

```
lrg = Table[rn[α, k, x], {k, 0, nt}]
```

```
{0.785398, -0.00831063, 0.10601, 0.00515443, -0.0383694, -0.00404563,
 0.0209097, 0.00343045, -0.0134837, -0.00302289, 0.00952576, 0.00272522, -0.007121,
 -0.00249378, 0.00552853, 0.00230577, -0.00440754, -0.00214799, 0.00358162,
 0.00201219, -0.00295122, -0.00189296, 0.00245633, 0.00178654, -0.00205884,
 -0.00169027, 0.00173352, 0.00160219, -0.00146304, -0.00152085, 0.00123515}
```

■ **Partial sums**

```
lps =  FoldList[Plus, lrg[[1]], Drop[lrg, 1]]
```

```
{0.785398, 0.777088, 0.883098, 0.888252, 0.849883, 0.845837, 0.866747,
 0.870177, 0.856694, 0.853671, 0.863196, 0.865922, 0.858801, 0.856307, 0.861835,
 0.864141, 0.859734, 0.857586, 0.861167, 0.863179, 0.860228, 0.858335, 0.860792,
 0.862578, 0.860519, 0.858829, 0.860563, 0.862165, 0.860702, 0.859181, 0.860416}
```

```
lps - ff[x]
```

```
{-0.0751934, -0.083504, 0.0225062, 0.0276606, -0.0107088, -0.0147544, 0.00615523,
 0.00958568, -0.00389801, -0.0069209, 0.00260486, 0.00533008, -0.00179092,
 -0.0042847, 0.00124382, 0.00354959, -0.000857943, -0.00300593, 0.000575685,
 0.00258788, -0.000363336, -0.00225629, 0.000200033, 0.00198657, -0.000072274,
 -0.00176254, -0.0000290179, 0.00157318, 0.000110135, -0.00141071, -0.000175562}
```

■ **Terms of Shanks Sequence**

```
lsh = Table[shanks[lps, k], {k, nt}]
```

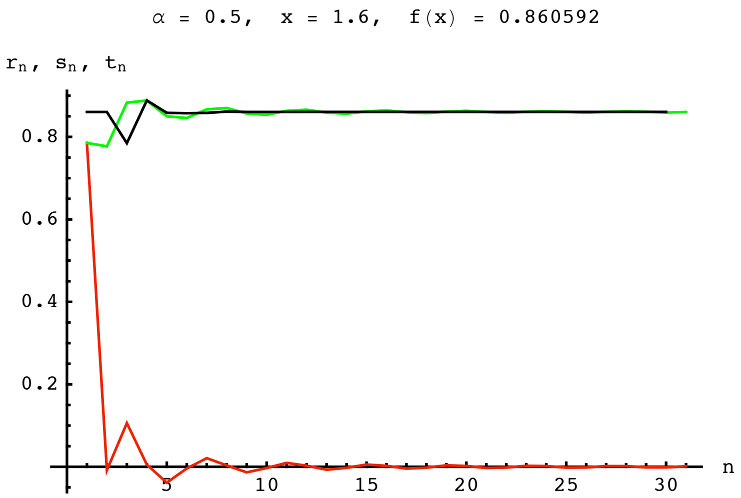
```
{0.860416, 0.860416, 0.784794, 0.888516, 0.858284, 0.857486, 0.858156,
 0.861343, 0.86053, 0.860489, 0.860521, 0.860614, 0.86059, 0.860588, 0.860589,
 0.860592, 0.860591, 0.860591, 0.860591, 0.860592, 0.860592, 0.860592,
 0.860592, 0.860592, 0.860592, 0.860592, 0.860592, 0.860592, 0.860592, 0.860592}
```

```
lsh - ff[x]
```

{-0.000175562, -0.000175562, -0.0757975, 0.027924, -0.00230777, -0.00310533, -0.00243569, 0.000751145, -0.0000613005, -0.000102708, -0.0000707994, 0.000022663, -1.52903×10^{-6}, -3.32664×10^{-6}, -2.04135×10^{-6}, 7.03466×10^{-7}, -3.54073×10^{-8}, -1.07015×10^{-7}, -5.90738×10^{-8}, 2.20956×10^{-8}, -7.21991×10^{-10}, -3.4315×10^{-9}, -1.71971×10^{-9}, 6.98342×10^{-10}, -1.0774×10^{-11}, -1.09833×10^{-10}, -5.03834×10^{-11}, 2.21539×10^{-11}, 2.22045×10^{-14}, -3.51097×10^{-12}}

- **Plots**

```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]

ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0]]

ph = ListPlot[lsh, Joined → True]

Show[pr, ps, ph, AxesLabel → {"n", Row[{rn, "， ", sn, "， ", tn}]},
 PlotLabel → Row[{"α = ", α, ",   x = ", x, ",   f(x) = ", ff[x], "\n\n"}]]
```



3.2 Acceleration of convergence rate for a very slowly converging Fourier series

$\alpha = 0.5$ no longer corresponds to a Fourier Series

```
nt = 30; (* number of terms *)
α = -.49; x = 1.6;
 ff[xx_] = fg[α, xx]; ff[x]
Integrate[ff[xx]^2, {xx, -π, π}]
```

1.15851

0.

- **Terms of the series**

```
lrg = Table[rn[α, k, x], {k, 0, nt}]
```

{1.54942, 0.0267849, -0.658841, -0.0473229, 0.465597, 0.0610071, -0.376761, -0.0718753, 0.32201, 0.0810345, -0.283193, -0.0889741, 0.253258, 0.0959583, -0.228831, -0.102147, 0.208073, 0.107645, -0.189893, -0.112525, 0.173598, 0.116837, -0.158729, -0.120622, 0.14497, 0.12391, -0.132096, -0.126723, 0.119946, 0.129081, -0.108401}

- **Partial sums**

```
lps = FoldList[Plus, lrg[[1]], Drop[lrg, 1]]
```

{1.54942, 1.57621, 0.917366, 0.870043, 1.33564, 1.39665, 1.01989, 0.948011, 1.27002, 1.35106, 1.06786, 0.978888, 1.23215, 1.3281, 1.09927, 0.997127, 1.2052, 1.31285, 1.12295, 1.01043, 1.18403, 1.30086, 1.14213, 1.02151, 1.16648, 1.29039, 1.15829, 1.03157, 1.15152, 1.2806, 1.1722}

```
lps - ff[x]
```

{0.390914, 0.417699, -0.241141, -0.288464, 0.177133, 0.23814, -0.138621, -0.210497, 0.111513, 0.192548, -0.0906451, -0.179619, 0.073639, 0.169597, -0.0592335, -0.161381, 0.0466924, 0.154338, -0.0355551, -0.14808, 0.0255181, 0.142356, -0.0163734, -0.136996, 0.00797391, 0.131883, -0.000212787, -0.126935, -0.00698898, 0.122092, 0.0136909}

- **Terms of Shanks Sequence**

```
lsh = Table[shanks[lps, k], {k, nt}]
```

{1.1722, 1.1722, 1.55047, 0.866381, 1.16953, 1.17222, 1.16988, 1.15394, 1.15881, 1.15897, 1.15884, 1.15839, 1.15852, 1.15852, 1.15852, 1.1585, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851, 1.15851}

```
lsh - ff[x]
```

{0.0136909, 0.0136909, 0.391961, -0.292127, 0.0110221, 0.0137132, 0.0113708, -0.00456709, 0.00030459, 0.000461881, 0.000336653, -0.000121708, 7.8533×10^{-6}, 0.0000150712, 9.75934×10^{-6}, -3.59443×10^{-6}, 1.89657×10^{-7}, 4.86252×10^{-7}, 2.82624×10^{-7}, -1.10025×10^{-7}, 4.16212×10^{-9}, 1.56111×10^{-8}, 8.21941×10^{-9}, -3.42446×10^{-9}, 7.5431×10^{-11}, 4.99875×10^{-10}, 2.40403×10^{-10}, -1.07563×10^{-10}, 6.86118×10^{-13}, 1.59819×10^{-11}}

- **Plots**

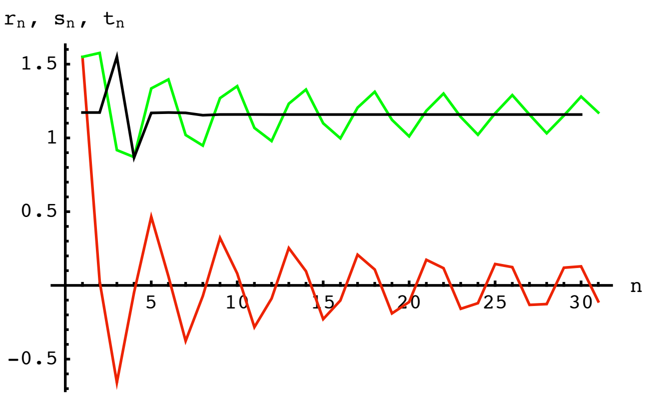
```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]
```

```
ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0]]
```

```
ph = ListPlot[lsh, Joined → True]
```

```
Show[pr, ps, ph, AxesLabel → {"n", Row[{rn, ",  ", sn, ",  ", tn}]},
 PlotLabel → Row[{"α = ", α, ",   x = ", x, ",   f(x) = ", ff[x], "\n\n"}]]
```

α = -0.49,   x = 1.6,   f(x) = 1.15851



■ 3.3 Acceleration of convergence rate for a very slowly converging trigonometric series

```
nt = 30; (* number of terms *)
α = -.5; x = 1.6;
ff[xx_] = fg[α, xx]; ff[x]
```

```
1.16199
```

■ Terms of the series

```
lrg = Table[rn[α, k, x], {k, 0, nt}]
```

```
{1.5708, 0.027909, -0.69084, -0.0498125, 0.491452, 0.0645346, -0.399256, -0.0762819,
 0.342201, 0.0862156, -0.301613, -0.0948505, 0.270218, 0.102465, -0.244527,
 -0.109229, 0.22264, 0.115251, -0.203425, -0.120608, 0.186164, 0.125355, -0.17038,
 -0.129533, 0.155745, 0.133174, -0.142028, -0.136302, 0.12906, 0.138937, -0.116718}
```

■ Partial sums

```
lps = FoldList[Plus, lrg[[1]], Drop[lrg, 1]]
```

```
{1.5708, 1.59871, 0.907865, 0.858053, 1.3495, 1.41404, 1.01478, 0.938501, 1.2807, 1.36692, 1.0653,
 0.970454, 1.24067, 1.34314, 1.09861, 0.989381, 1.21202, 1.32727, 1.12385, 1.00324, 1.1894,
 1.31476, 1.14438, 1.01484, 1.17059, 1.30376, 1.16174, 1.02543, 1.15449, 1.29343, 1.17671}
```

```
lps - ff[x]
```

```
{0.408805, 0.436714, -0.254126, -0.303939, 0.187513, 0.252048, -0.147209, -0.223491,
 0.11871, 0.204926, -0.096687, -0.191538, 0.07868, 0.181145, -0.0633821, -0.172611,
 0.0500297, 0.165281, -0.0381444, -0.158753, 0.027411, 0.152766, -0.0176134, -0.147147,
 0.00859854, 0.141772, -0.000255656, -0.136557, -0.00749735, 0.13144, 0.0147222}
```

■ Terms of Shanks Sequence

```
lsh = Table[shanks[lps, k], {k, nt}]
```

```
{1.17671, 1.17671, 1.57188, 0.854182, 1.17341, 1.17619, 1.17377, 1.15725, 1.16231, 1.16247,
 1.16234, 1.16187, 1.162, 1.16201, 1.162, 1.16199, 1.16199, 1.16199, 1.16199, 1.16199,
 1.16199, 1.16199, 1.16199, 1.16199, 1.16199, 1.16199, 1.16199, 1.16199, 1.16199, 1.16199}
```

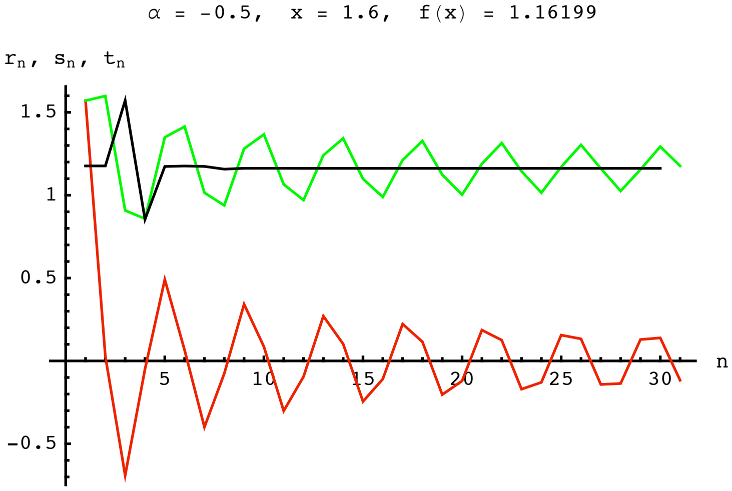
```
lsh - ff[x]
```

$\{0.0147222, 0.0147222, 0.409889, -0.30781, 0.0114175, 0.0141959, 0.0117768,$
$-0.00474423, 0.000315251, 0.000477642, 0.000348318, -0.000126094, 8.1267\times10^{-6},$
$0.0000155794, 0.0000100934, -3.72012\times10^{-6}, 1.96276\times10^{-7}, 5.02536\times10^{-7}, 2.92227\times10^{-7},$
$-1.13812\times10^{-7}, 4.30906\times10^{-9}, 1.61315\times10^{-8}, 8.49728\times10^{-9}, -3.5412\times10^{-9}, 7.81812\times10^{-11},$
$5.16485\times10^{-10}, 2.48497\times10^{-10}, -1.11209\times10^{-10}, 7.13873\times10^{-13}, 1.6509\times10^{-11}\}$

■ Plots

```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]

ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0]]

ph = ListPlot[lsh, Joined → True]
```

```
Show[pr, ps, ph, AxesLabel → {"n", Row[{rn, ", ", sn, ", ", tn}]},
 PlotLabel → Row[{"α = ", α, ",   x = ", x, ",   f(x) = ", ff[x], "\n\n"}]]
```



■ 3.4 Inducing convergence in a diverging trigonometric series

```
nt = 30; (* number of terms *)
α = - 1.5; x =  1.6;
 ff[xx_] = fg[α, xx]; ff[x]
```

1.56895

■ Terms of the series

```
lrg = Table[rn[α, k, x], {k, 0, nt}]
```

{0, -0.0820227, 4.18511, 0.457849, -6.05906, -0.99822, 7.42932,
 1.659, -8.51704, -2.41661, 9.40155, 3.25452, -10.1205, -4.15953, 10.6946,
 5.12033, -11.1362, -6.12674, 11.4531, 7.1693, -11.6509, -8.23909, 11.7336,
 9.32755, -11.7043, -10.4264, 11.5658, 11.5277, -11.3207, -12.6236, 10.9714}

■ Partial sums

```
lps =  FoldList[Plus, lrg[[1]], Drop[lrg, 1]]
```

{0, -0.0820227, 4.10309, 4.56094, -1.49811, -2.49633, 4.93299,
 6.59199, -1.92505, -4.34166, 5.05989, 8.31441, -1.80613, -5.96566, 4.72896,
 9.84929, -1.28687, -7.41361, 4.03951, 11.2088, -0.442119, -8.68121, 3.05239,
 12.3799, 0.675628, -9.75079, 1.81502, 13.3427, 2.02204, -10.6015, 0.369897}

```
lps - ff[x]
```

{-1.56895, -1.65097, 2.53414, 2.99199, -3.06706, -4.06528, 3.36404,
 5.02304, -3.494, -5.91061, 3.49094, 6.74546, -3.37508, -7.53461, 3.16001,
 8.28034, -2.85582, -8.98256, 2.47056, 9.63986, -2.01107, -10.2502, 1.48344,
 10.811, -0.893321, -11.3197, 0.246073, 11.7738, 0.453093, -12.1705, -1.19905}

■ Terms of Shanks Sequence

```
lsh = Table[shanks[lps, k], {k, nt}]
```

{0.369897, 0.369897, -0.00157663, 4.61718, 1.72986, 1.75768, 1.73286, 1.47818, 1.57179, 1.57294,
 1.572, 1.56769, 1.56901, 1.56906, 1.56903, 1.56892, 1.56895, 1.56895, 1.56895, 1.56895,
 1.56895, 1.56895, 1.56895, 1.56895, 1.56895, 1.56895, 1.56895, 1.56895, 1.56895, 1.56895}

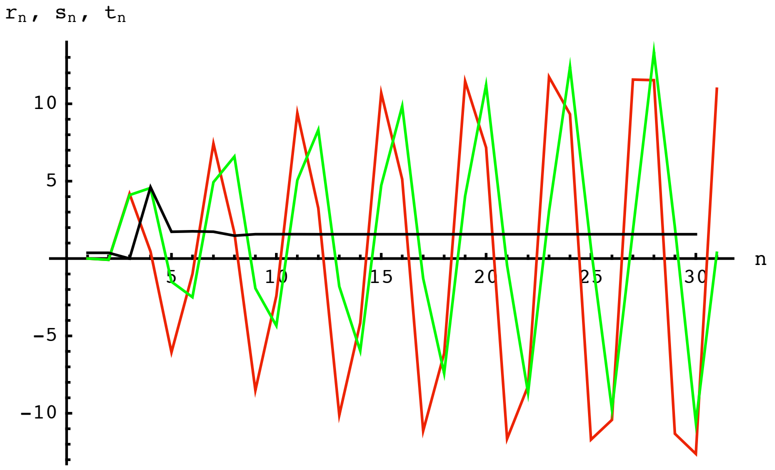
```
lsh - ff[x]
```

{-1.19905, -1.19905, -1.57053, 3.04823, 0.16091, 0.188734, 0.163911, -0.090767, 0.00283947, 0.00399363, 0.00305343, -0.00125608, 0.0000658506, 0.000115402, 0.0000781731, -0.0000306663, 1.53414×10^{-6}, 3.50906×10^{-6}, 2.12811×10^{-6}, -8.57943×10^{-7}, 3.38265×10^{-8}, 1.08703×10^{-7}, 5.95841×10^{-8}, -2.53479×10^{-8}, 6.5483×10^{-10}, 3.39807×10^{-9}, 1.69795×10^{-9}, -7.69618×10^{-10}, 8.57647×10^{-12}, 1.07209×10^{-10}}

- **Plots**

```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]

ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0]]

ph = ListPlot[lsh, Joined → True]

Show[pr, ps, ph, AxesLabel → {"n", Row[{rn, ", ", sn, ", ", tn}]},
 PlotLabel → Row[{"α = ", α, ",   x = ", x, ",   f(x) = ", ff[x], "\n\n"}]]
```

α = -1.5,   x = 1.6,   f(x) = 1.56895



4. Point near to the end of the interval

Die Shankstransformation konvergiert schlecht in der Naehe von $|x| = \pi$! Sehr genaue Kontrollen sind noetig, damit man verlaessliche und genaue
Resultate erhaelt.

■ 4.1 Acceleration of convergence rate for a slowly converging Fourier series

```
nt = 250; (* number of terms *)
α = .5; x = 3.1;
ff[xx_] = fg[α, xx]; ff[x]
NIntegrate[ff[xx]^2, {xx, -π, π}]
```

0.162183

4.18879

- **Terms of the series**

```
{tr, lrg} = Table[rn[α, k, x], {k, 0, nt}] // Timing;
tr
```

0.07 Second

```
lrg[[Range[230, 250]]]
```

```
{0.0000913478, 0.000090296, 0.0000891014, 0.0000877679, 0.0000862994, 0.0000847002, 0.0000829747,
 0.0000811273, 0.0000791627, 0.0000770857, 0.0000749012, 0.0000726142, 0.0000702298, 0.0000677533,
 0.00006519, 0.0000625452, 0.0000598245, 0.0000570332, 0.000054177, 0.0000512615, 0.0000482923}
```

- **Partial sums**

```
{tp, lps} = FoldList[Plus, lrg[[1]], Drop[lrg, 1]] // Timing;
tp
```

```
0. Second
```

```
lps[[Range[230, 250]]]
```

```
{0.162123, 0.162213, 0.162302, 0.16239, 0.162476, 0.162561, 0.162644,
 0.162725, 0.162804, 0.162881, 0.162956, 0.163029, 0.163099, 0.163167,
 0.163232, 0.163294, 0.163354, 0.163411, 0.163465, 0.163517, 0.163565}
```

```
lps - ff[x]
```

```
Min[Abs[%]]
```

```
0.0000294978
```

- **Terms of Shanks Sequence**

```
{ts, lsh} = Table[shanks[lps, k], {k, nt}] // Timing;
ts
```

```
66.09 Second
```

```
lsh - ff[x]
```

```
Min[Abs[%]]
```

2.99867×10^{-7}

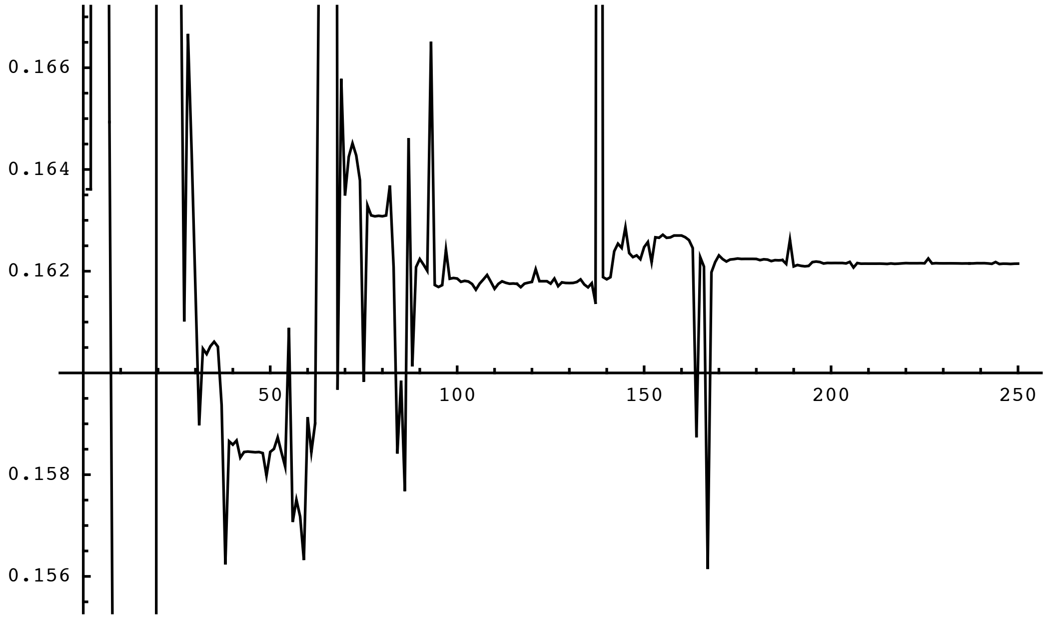
- **Plots**

```
ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0], PlotRange → All, ImageSize → 700]
```

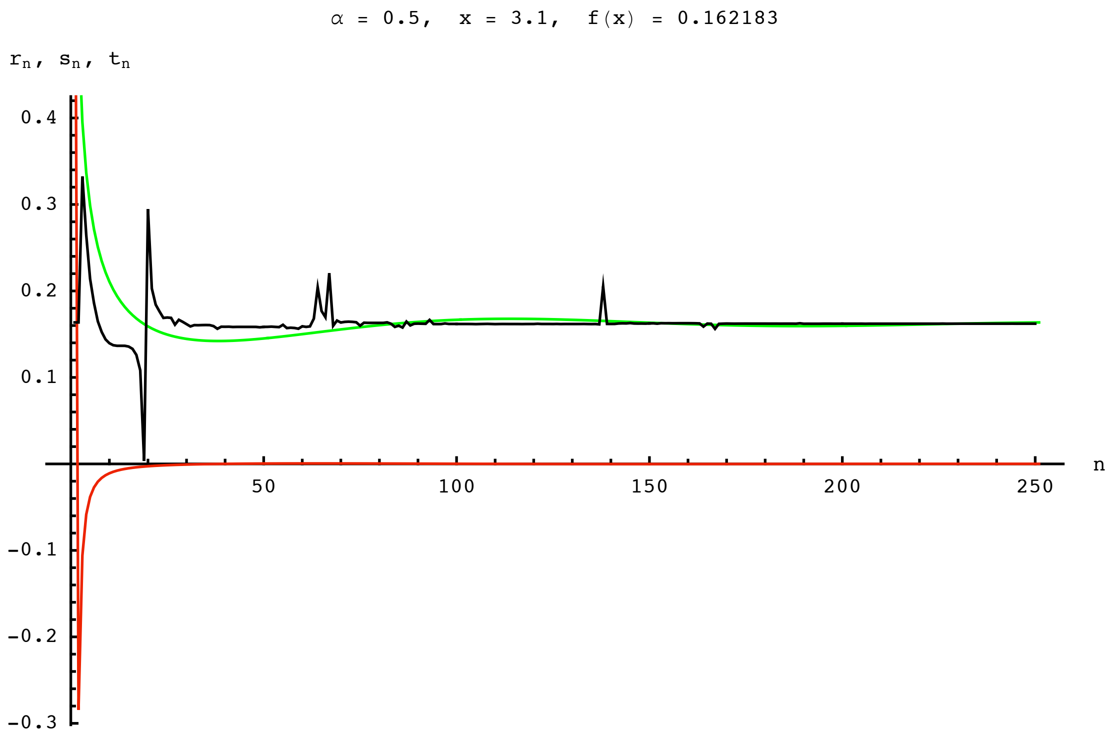
```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]
```

```
ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0],
  Epilog → {Hue[0], Line[{{1, ff[x]}, {nt, ff[x]}}]}]
```

```
ph = ListPlot[lsh, Joined → True]
```



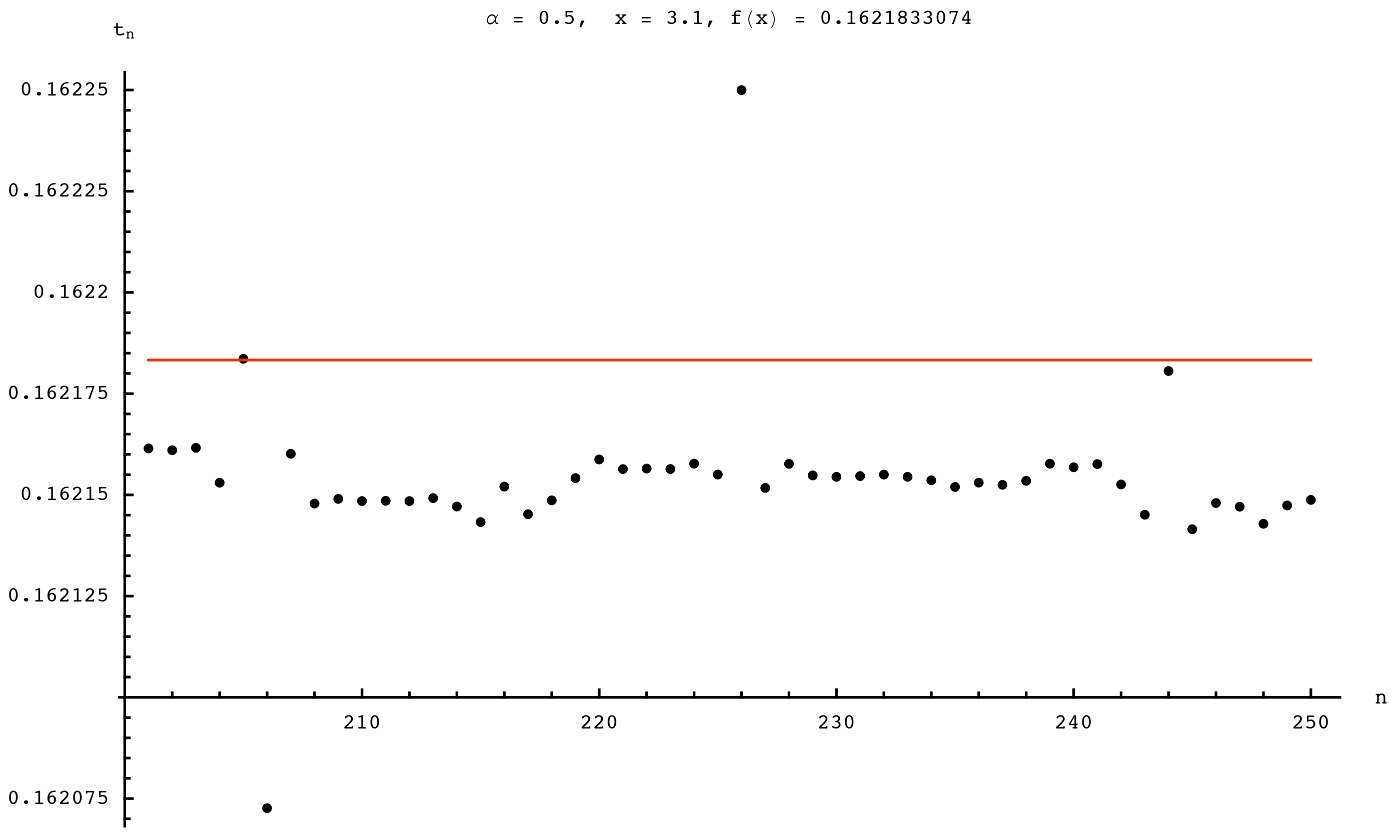

```
p411 = Show[pr, ps, ph, AxesLabel → {"n", Row[{rn, ", ", sn, ", ", tn}]},
  PlotLabel → Row[{"α = ", α, ",  x = ", x, ",  f(x) = ", ff[x], "\n\n"}]]
```




```
nr = 50;
gen = 10;
var = 10^(-10);
pn = Range[nt - nr + 1, nt]

lshr = SetPrecision[Take[lsh, -nr], 10];
lnsh = Transpose[{pn, lshr}]

tiy = SetPrecision[{ff[x]}, gen]
```

{0.1621833074}

```
var = 1/10^7;
p412 = ListPlot[lnsh,
  PlotLabel → Row[{"α = ", α, ",  x = ", x, ", f(x) = ", SetPrecision[ff[x], gen], "\n"}],
  PlotRange → All, AxesLabel → {n, t_n},
  Epilog → {Hue[0], Line[{{pn[[1]], ff[x]}, {pn[[-1]], ff[x]}}]}, ImageSize → 650]
```




```
Show[GraphicsRow[{p411, p412}], ImageSize → 650]
```

■ 4.2 Acceleration of convergence rate for a very slowly converging Fourier series

α = - 0.5 no longer corresponds to a Fourier Series

```
nt = 300; (* number of terms *)
α = -.49; x = 3.1;
 ff[xx_] = fg[α, xx]; ff[x]
Integrate[ff[xx]^2, {xx, -π, π}]
```

```
5.94558
```

```
0.
```

■ **Terms of the series**

```
lrg = Table[rn[α, k, x], {k, 0, nt}];
```

■ **Partial sums**

```
lps = FoldList[Plus, lrg[[1]], Drop[lrg, 1]];
```

```
lps - ff[x]
```

```
Min[Abs[%]]
```

```
0.0211968
```

- **Terms of Shanks Sequence**

```
lsh = Table[shanks[lps, k], {k, nt}]
```

```
lsh - ff[x]
```

```
Min[Abs[%]]
```

```
0.0000182948
```

- **Plots**

```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]
```

```
ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0]]
```

```
ph = ListPlot[lsh, Joined → True]
```

```
p421 = Show[pr, ps, ph, AxesLabel → {"n", Row[{rn, ",  ", sn, ",  ", tn}]},
  PlotLabel → Row[{"α = ", α, ",   x = ", x, ",   f(x) = ", ff[x], "\n\n"}]]
```

```
nr = 50;
gen = 10;
var = 10^(-10);
pn = Range[nt - nr + 1, nt]
```

```
lshr = SetPrecision[Take[lsh, -nr], 10];
lnsh = Transpose[{pn, lshr}]
```

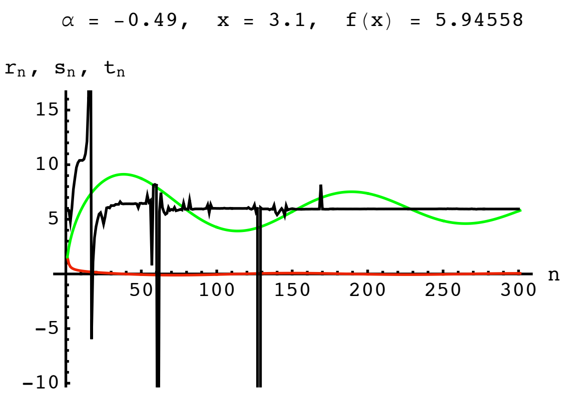
```
tiy = SetPrecision[{ff[x]}, gen]
```

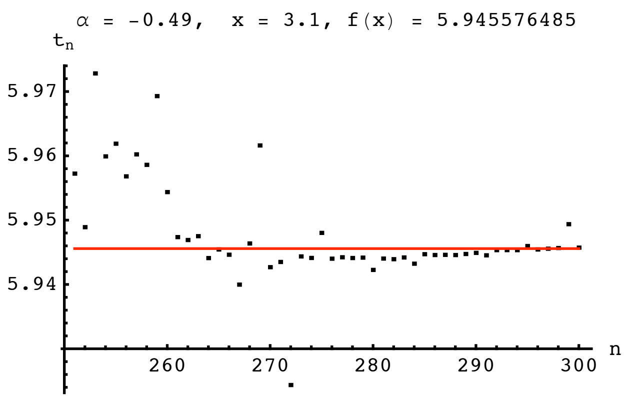
```
{5.945576485}
```

$$\text{var} = \frac{1}{10^7};$$

```
p422 = ListPlot[lnsh,
  PlotLabel → Row[{"α = ", α, ",   x = ", x, ", f(x) = ", SetPrecision[ff[x], gen], "\n"}],
  PlotRange → All, AxesLabel → {n, tn},
  Epilog → {Hue[0], Line[{{pn[[1]], ff[x]}, {pn[[-1]], ff[x]}}]}, ImageSize → 650]
```

```
Show[GraphicsRow[{p421, p422}], ImageSize → 650]
```





Large number of terms

Terms

```
{trl, lrgl} = Table[rn[α, k, x], {k, 0, 30000}] // Timing;
trl
```

```
General::spell1 :
 Possible spelling error: new symbol name "lrgl" is similar to existing symbol "lrg". More…
```

```
7.18 Second
```

Partial sums

```
{tpl, lpsl} =  FoldList[Plus, lrgl[[1]], Drop[lrgl, 1]] // Timing;
tpl
```

```
General::spell1 :
 Possible spelling error: new symbol name "lpsl" is similar to existing symbol "lps". More…
```

```
0.05 Second
```

```
dps = lpsl - ff[x];
```

```
Min[dps // Abs]
```

```
0.0000113871
```

```
Print["Time for performing Shanks trf with   ", nt, "   terms:   ", ts]
Print["Time for computing partial sums with ", 30000, " terms: ", trl + tpl]
lsh[[-1]]
lpsl[[-1]]
ff[x]
```

```
Time for performing Shanks trf with   300   terms:   64.09 Second

Time for pcomputing partial sums with 30000 terms: 7.23 Second
```

```
5.94573

5.87886

5.94558
```

In this case the result of the Shanks transformation with 300 terms is more precise than the partial sum with 30000 terms.
But it was found that just with 250 terms one gets a less precise result.

4.3 Acceleration of convergence rate for a very slowly converging trigonometric series

```
nt = 300; (* number of terms *)
α = -.5; x = 3.1;
 ff[xx_] = fg[α, xx]; ff[x]
```

```
6.16586
```

Terms of the series

```
lrg = Table[rn[α, k, x], {k, 0, nt}];
```

Partial sums

```
lps =  FoldList[Plus, lrg[[1]], Drop[lrg, 1]];
```

```
lps - ff[x];
```

```
Min[Abs[%]]
```

```
0.0174533
```

```
%%[[-Range[20]]]
```

```
{-0.147613, -0.205114, -0.262449, -0.319517, -0.37622, -0.432458,
 -0.488131, -0.543143, -0.597394, -0.65079, -0.703235, -0.754634, -0.804895,
 -0.853928, -0.901642, -0.947952, -0.992772, -1.03602, -1.07761, -1.11748}
```

- **Terms of Shanks Sequence**

```
lsh = Table[shanks[lps, k], {k, nt}]
```

```
lsh - ff[x]
```

```
Min[Abs[%]]
```

```
0.0000639156
```

```
%%[[-Range[20]]]
```

```
{0.0000639156, -0.000314166, 0.00232767, 0.0015516, 0.00251117, 0.000677725, 0.00106865,
 0.00100628, 0.00289313, 0.00122829, 0.00257051, 0.00099057, 0.00105308, 0.00159442,
 0.000283831, -0.000331206, -0.0165515, -0.000759875, -0.000557795, -0.000647175}
```

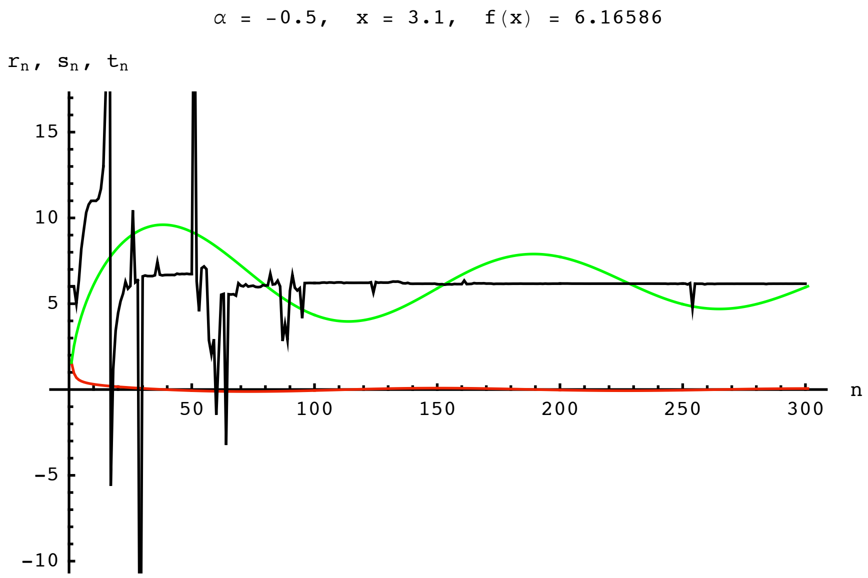
- **Plots**

```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]
```

```
ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0]]
```

```
ph = ListPlot[lsh, Joined → True]
```

```
p441 = Show[pr, ps, ph, AxesLabel → {"n", Row[{rn, ", ", sn, ", ", tn}]},
  PlotLabel → Row[{"α = ", α, ",  x = ", x, ",  f(x) = ", ff[x], "\n\n"}]]
```



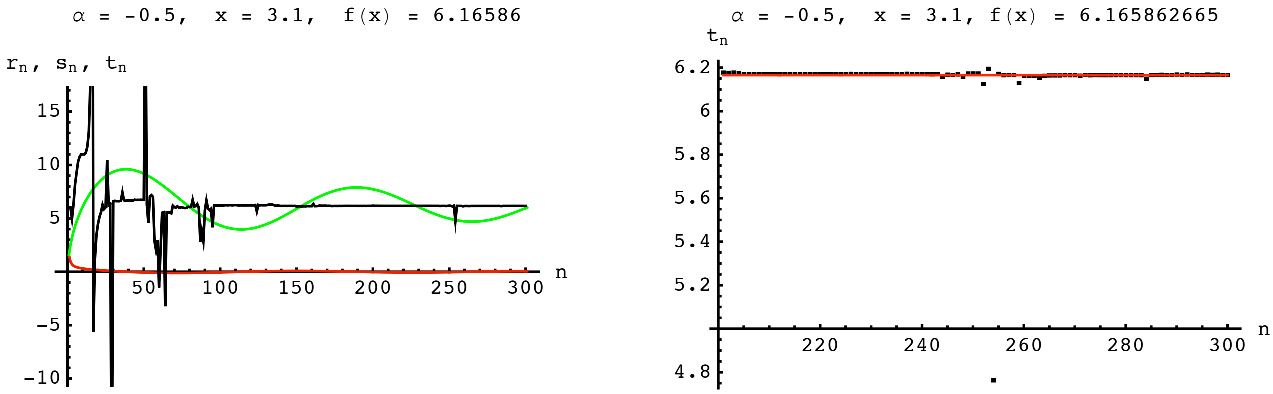

```
nr = 100;
gen = 10;
var = 10^(-10);
pn = Range[nt - nr + 1, nt]
```

```
lshr = SetPrecision[Take[lsh, -nr], 10];
lnsh = Transpose[{pn, lshr}]
```

```
tiy = SetPrecision[{ff[x]}, gen]
```

{6.165862665}

```
var = 1/10^7;
p442 = ListPlot[lnsh,
  PlotLabel → Row[{"α = ", α, ",  x = ", x, ", f(x) = ", SetPrecision[ff[x], gen], "\n"}],
  PlotRange → All, AxesLabel → {n, t_n},
  Epilog → {Hue[0], Line[{{pn[[1]], ff[x]}, {pn[[-1]], ff[x]}}]}, ImageSize → 650]
Show[GraphicsRow[{p441, p442}], ImageSize → 650]
```



- **Large number of terms**

- **Terms**

```
{trl, lrgl} = Table[rn[α, k, x], {k, 0, 30000}] // Timing;
trl
```

7.04 Second

- **Partial sums**

```
{tpl, lpsl} =  FoldList[Plus, lrgl[[1]], Drop[lrgl, 1]] // Timing;
tpl
```

0.04 Second

```
dps = lpsl - ff[x];
Min[dps // Abs]
```

0.0000110063

```
Print["Time for performing Shanks trf with   ", nt, "   terms:   ", ts]
Print["Time for computing partial sums with ", 30000, " terms: ", trl + tpl]
lsh[[-Range[20]]]
lpsl[[-1]]
ff[x]
```

```
Time for performing Shanks trf with   300   terms:   70.7 Second

Time for computing partial sums with 30 000 terms: 7.08 Second

{6.16593, 6.16555, 6.16819, 6.16741, 6.16837, 6.16654, 6.16693, 6.16687, 6.16876, 6.16709,
 6.16843, 6.16685, 6.16692, 6.16746, 6.16615, 6.16553, 6.14931, 6.1651, 6.1653, 6.16522}

6.08889

6.16586
```

In this case the result of the Shanks transformation with 300 terms is more precise than the partial sum with 30000 terms. But one can see the scattering of the results in the drawing.

▪ 4.4 Inducing convergence in a diverging trigonometric series

```
nt = 300; (* number of terms *)
α = - 1.5; x = 3.1;
ff[xx_] = fg[α, xx]; ff[x]

234.413
```

▪ **Terms of the series**

```
lrg = Table[rn[α, k, x], {k, 0, nt}];
```

▪ **Partial sums**

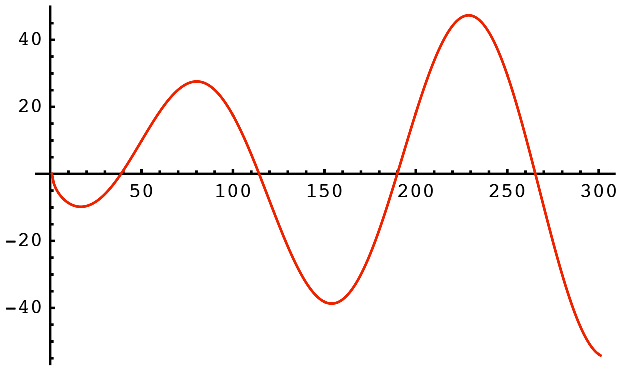
```
lps = FoldList[Plus, lrg[[1]], Drop[lrg, 1]];
```

▪ **Terms of Shanks Sequence**

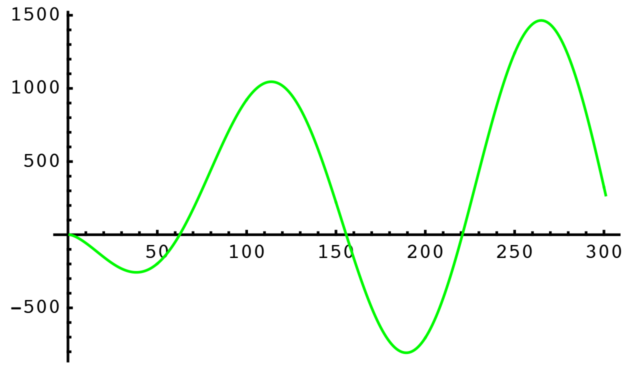
```
lsh = Table[shanks[lps, k], {k, nt}]

lsh - ff[x]

Min[Abs[%]]

0.0019365

%%[[-Range[20]]]

{-0.140921, -0.137462, -0.210296, -0.0907356, -0.141379, -0.102504,
 -0.0485643, -0.0211363, -0.0998544, 0.0019365, -0.0345472, -0.08276, -0.0612618,
 -0.0683973, 0.111715, -0.0866674, -0.108273, -0.124511, -0.103749, -0.085531}
```

▪ **Plots**

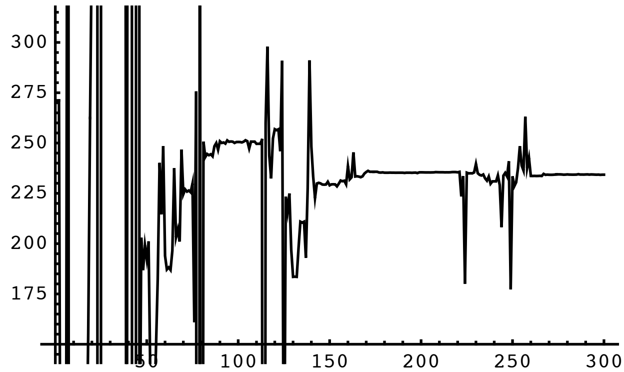
```
pr = ListPlot[lrg, Joined → True, PlotStyle → RGBColor[0.9`, 0, 0]]
```



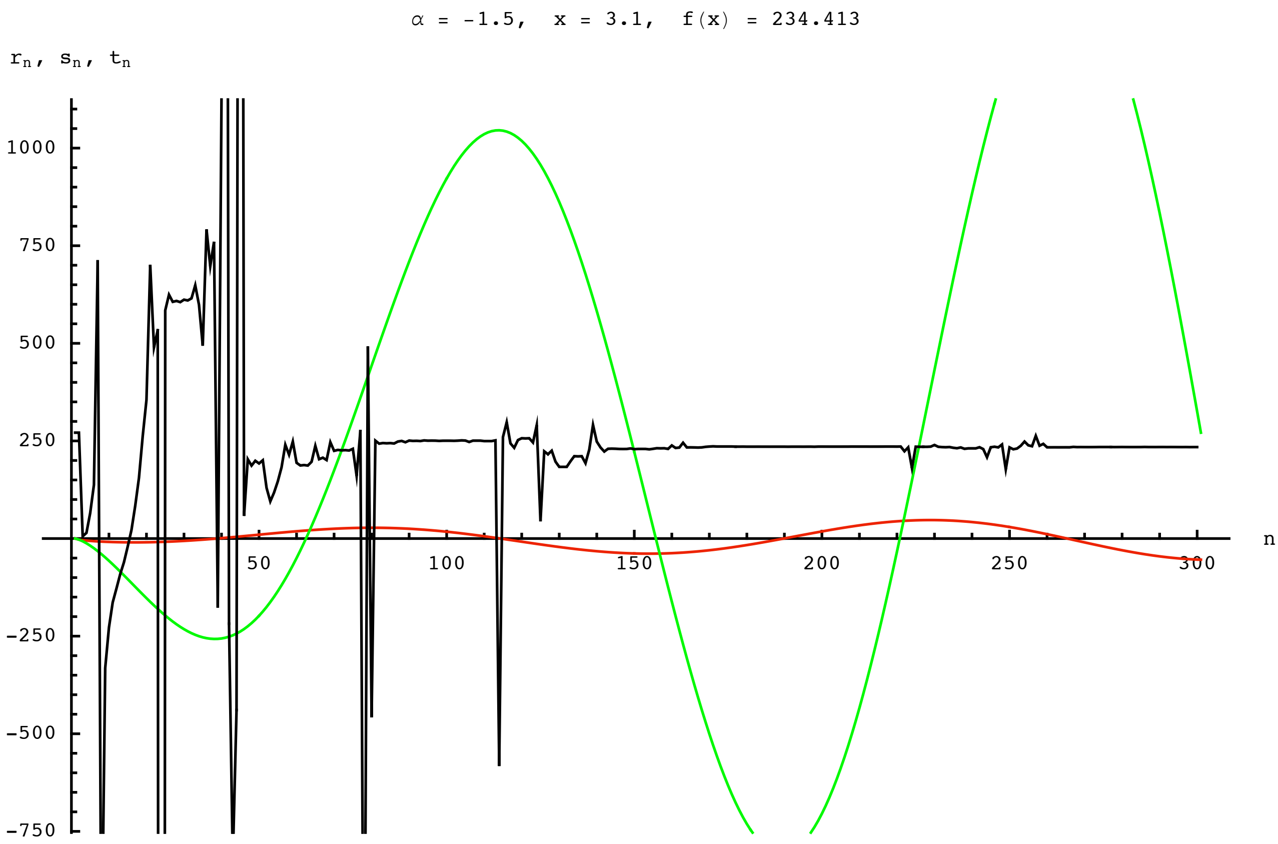
```
ps = ListPlot[lps, Joined → True, PlotStyle → RGBColor[0, 1, 0]]
```




```
ph = ListPlot[lsh, Joined → True]
```



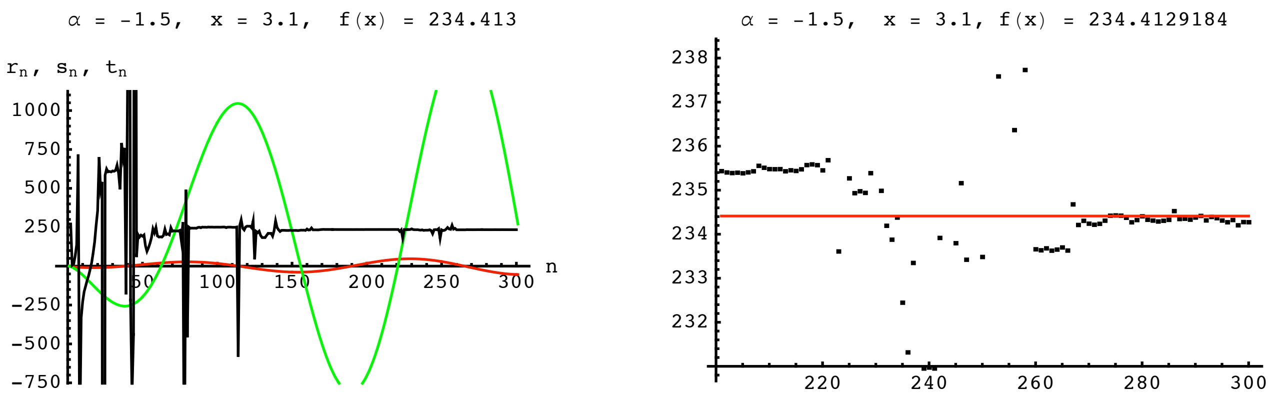

```
p451 = Show[pr, ps, ph, AxesLabel → {"n", Row[{rₙ, ", ", sₙ, ", ", tₙ}]},
  PlotLabel → Row[{"α = ", α, ",   x = ", x, ",   f(x) = ", ff[x], "\n\n"}]]
```



```
nr = 100;
gen = 10;
var = 10^(-10);
pn = Range[nt - nr + 1, nt]

lshr = SetPrecision[Take[lsh, -nr], 10];
lnsh = Transpose[{pn, lshr}]

tiy = SetPrecision[{ff[x], 1.56894895, 1.56894893, 1.56894892, 1.56894894,
   1.56894891, 1.56894897, 1.56894896, 1.56894894, 1.56894898, 1.56894899, 1.568949}, gen]
```

$$var = \frac{1}{10^8};$$

```
p452 =
 ListPlot[lnsh, PlotLabel → Row[{"α = ", α, ",   x = ", x, ", f(x) = ", SetPrecision[ff[x], gen]}],
  Epilog → {Hue[0], Line[{{pn[[1]], ff[x]}, {pn[[-1]], ff[x]}}]}, ImageSize → 700]

Show[GraphicsRow[{p451, p452}], ImageSize → 650]
```



As the series diverges the partial sums give no result. The Shanks transformation still leads to an approximate result.