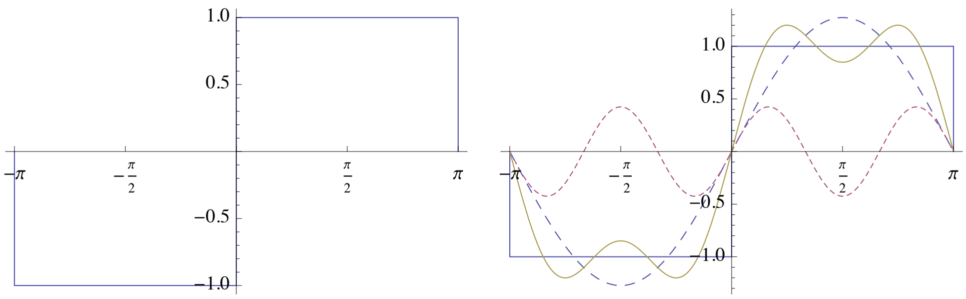
■ 9.1.1.1 Fourier Series and the Gibb's Phenomenon. Fejer Sum

In 1899 the Canadian theoretical phycisist W. Gibbs found a peculiar behaviour of Fourier series representing discontinuous functions. To show this peculiarity, called Gibb's phenomenon, the Fourier series belonging to the periodic step function displayed below in the left picture is summed.

```
me = {{-π, 0}, {-π, -1}, {0, -1}, {0, 1}, {π, 1}, {π, 0}};
p1 = ListPlot[me, Joined → True, Ticks → {Pi Range[-1, 1, 1 / 2], Automatic}];

Clear[n,x];
yf[n_,x_] = (4/Pi) Sin[n x]/n;
sf[n_,x_] :=  Sum[ yf[i,x], {i,1,n,2}];
ssf  = Plot[ {sf[2,x], yf[3,x], sf[3,x]}, {x,Pi,-Pi},
 PlotStyle -> {Dashing[{.02}],Dashing[{.01}],Dashing[{}]},
 Ticks->{Pi Range[-1,1,1/2], Automatic}];

p2 = Show[p1, ssf];
Show[GraphicsRow[{p1, p2}], ImageSize → 450]
```

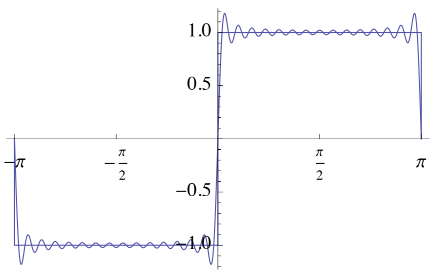


The picture at the right shows the first harmonic (curve with long dashes), the second harmonic (short dashes) and the sum of these two (continuous curve). The first harmonic overshootes the value 1 at $x = \pm\pi/2$. The second harmonic (over)compensates this effect. and leads to overshooting near $x = \pm\pi/4$ and $x = \pm 3\pi/4$. The next harmonic will compensate all these overshootings but will introduce new overshootings at points stlll nearer to the discontinuities of the step functions. At points in the interior of the intefvalls $(-\pi,0)$ and $(0,\pi)$ this wiggles will smooth out in the limit $N \to \infty$.

$$s(x) \quad = \quad \lim_{N\to\infty} \Sigma_{n=1}^{N}\, s_n(x) \quad = \quad \lim_{N\to\infty} \Sigma_{n=1}^{N}\, b_n \sin(n\,x)$$

But there remain steep narrow peaks approaching the points where the steps occur, as can be seen in the picture below.

```
Plot[Evaluate[sf[30,x]], {x,-Pi,Pi},
     Ticks->{Pi Range[-1,1,1/2], Automatic}];
Show[%,p1, ImageSize ->200]
```



Analytic analysis shows that these overshoots at the jumps remain in the limit $N \to \infty$, but the value of the limit $s(x)$ depends very sensitively upon the way the limits $N \to \infty$ and $x \to 0+$, for example, are taken. The most important point is: Common summation of Fourier series representing discontinuous functions leads to Gibb's phenomenon. This can be avoided by summing the Fourier series in a different way, namely by Fejer's arithmetic means:

$$s(x) \quad = \quad \lim_{N\text{->}\infty} \quad \Sigma_{n=1}{}^{N}\ \sigma_n\ (x),$$

$$\sigma_n(x) \ = n^{-1}\ \Sigma_{i=1}{}^{n}\ s_i(x) \ = n^{-1}\ \Sigma_{i=1}{}^{n}{}_{k=1}{}^{i}\ b_k \sin(k\ x)$$

```
sff[n_,x_] :=  Sum[ sf[i,x], {i,n}]/n ;
Plot[Evaluate[{sff[2,x], sf[3,x], sff[3,x]}], {x,Pi,-Pi},
 PlotStyle -> {Dashing[{.02}],Dashing[{.01}],Dashing[{}]},
 Ticks -> {Pi Range[-1,1,1/2],Automatic}];

ss3 = Show[p1, %];

Plot[Evaluate[sff[20,x]], {x,Pi,-Pi}];
ss0 = Show[p1, %, Ticks -> {Pi Range[-1,1,1/2],Automatic} ];
Show[GraphicsRow[{ss3,ss0}], ImageSize → 450]
```