Kapitel 8

Series and their sums

8.1 Convergence of series

In general, a series comprises an infinity of terms:

$$r_1 \; + \; r_2 \; + \; r_3 \; + ... \; = \sum_{n=1}^{\infty} r_n \tag{8.1}$$

The most important question is how to assign a reasonable value to this infinity of numbers or functions. The standard procedure is to form the partial sums:

$$s_N \; := \; \sum_{n=1}^{N} r_n \tag{8.2}$$

and to investigate the convergence of this sequence of numbers or functions:

$$s_1, s_2, s_3, s_4, ..., s_n, ... \tag{8.3}$$

If this sequence tends toward a limit s, then this is called the sum of the series (8.1):

$$\begin{gathered} s_1, s_2, s_3, s_4, ..., s_n, ... \;\; \rightarrow \;\; s \\ \sum_{n=1}^{\infty} r_n \; = \; s. \end{gathered} \tag{8.4}$$

A means to find out whether a series will tend toward such a limit is the
Ratio Test: If $\lim_{n\to\infty} |r_{n+1}/r_n| \; < \; 1$, the series $\sum_{n=1}^{\infty} r_n$ is absolutely convergent.
If $\lim_{n\to\infty} |r_{n+1}/r_n| \; > \; 1$, the series $\sum_{n=1}^{\infty} r_n$ is divergent. □
If $\lim_{n\to\infty} |r_{n+1}/r_n| \; = \; 1$, a further test must be applied. One such test is the following:

$$\left| \frac{r_n}{r_{n+1}} \right| \; = \; 1 + \frac{\mu}{n} + \frac{\omega_n}{n^2}, \quad \mu = \text{const.}, \;\; |\omega_n| < A \; \forall \; n, \tag{8.5}$$

(where μ is a constant and $|\omega_n|$ is less than a fixed number A for all n), the series $\sum_{n=1}^{\infty} r_n$ is convergent if $\mu > 1$ and divergent if $\mu \leq 1$.

8.2 Linear series transformatios for accelerating or inducing convergence

The definition (8.4) of the sum of a series is by no means the only possibility to assign a reasonable value to an infinite series. There are definitions of such a value, which may be applicable to a larger range of series. Of course, they must lead to the same value if applied to series convergent in the sense of the preceeding section. But they may assign a reasonable finite value to a series, which is divergent according to the criteria applied in the preceeding section. Series, where this applies, will be termed **summable**.

A few such methods will be described now:

8.2.1 Hölder means

A new sequence is generated from sequence (8.3) by taking arithmetic averages of the partial sums:

$$h_n^{(1)} \ := \ \frac{s_1 + s_2 + ... + s_n}{n} \tag{8.6}$$

If this sequence tends toward a finite limit:

$$\lim_{n\to\infty} h_n^{(1)} = s$$

the sequence $\{h_n^{(1)}\}$ is called H_1-limitable, and s its H_1-sum.

As an example take the geometric series

$$1 \ + \ q \ + \ q^2 \ + \ q^3 \ + ... \ = \ \frac{1}{1-q} \qquad \text{für} \qquad |q| \ < \ 1, \tag{8.7}$$

whose circle of convergence is $|q| = 1$. For $q = -1$ one gets in a formal way:

$$1 \ - \ 1 \ + \ 1 \ - 1 \ ... \ = \ \frac{1}{2} \ = \ \sum_{n=1}^{\infty} r_n \quad \text{with} \quad r_n \ = \ (-1)^{(n-1)}.$$

The sequence of partial sums is:

$$s_N \ = \ \sum_{n=1}^{N} r_n \ = \ \sum_{n=1}^{N} (-1)^{n-1} \ = \ \frac{1}{2}[-1 \ + \ (-1)^N], \qquad \{s_N\} \ = \ \{1.0, 1, 0, ..., 1, 0, ...\}. \tag{8.8}$$

The arithmetic means of the partial sums, eq.(8.6), give:

$$h_n^{(1)} := \ \frac{s_1 + s_2 + ... + s_n}{n} \ = \ \frac{1}{2} \ + \ \frac{1-(-1)^n}{4n}. \tag{8.9}$$

The sequence of $h_n^{(1)}$'s converges towards the limit $\frac{1}{2}$. For $q = -1$ the geometric series is not convergent but H_1-limitable to the value $\frac{1}{2}$. The same result obtains in *Mathematica* under the command NSum[].

H_2-means use a new sequence obtained from (8.6) by applying again an arithmetic mean:

$$h_n^{(2)} \ := \ \frac{h_1^{(1)} + h_2^{(1)} + ... + h_n^{(1)}}{n}. \tag{8.10}$$

In similar manner still higher everages, H_r-means, are introduced.

8.2.2 Borel summation

8.3 Non-linear series tranformations for accelerating and inducing convergence. The Shanks transform

In the preceeding sections linear combinations of partial sums were considered. In some cases it is more efficient to find new sequences t_N from non-linear functions of the partial sums.

$$t_N \ := \ f(s_1, s_2, s_3, s_4, ..., s_N). \tag{8.11}$$

An example of such a nonlinear transform is the Shanks transform; it is calculated by the followng algorithm:

$$\epsilon_{-1}^{(n)} \ = \ 0, \quad \epsilon_0^{(n)} \ = \ s_n, \tag{8.12}$$

$$\epsilon_{k+1}^{(n)} \ = \ \epsilon_{k-1}^{(n+1)} \ + \ 1/[\epsilon_k^{(n+1)} \ - \ \epsilon_k^{(n)}]; \quad k, n = 0, 1, 2, ... \tag{8.13}$$

$$t_k(s_n) \ = \ \epsilon_{2k}^{(n)}. \tag{8.14}$$

This scheme involves two chains of indices. The elements with even subscript give the terms of the Shanks sequence; those with odd subscript are just auxiliary quantities.

This method is equivalent to Padé approximants. The use of this method implies some dangers. Examples are given in § 9.1.3 .

[8.1] Divergent series, Chap.12 in W.R. Gibbs: Computation in Modern Physics. World Scientific, 2 nd ed., 2003.

[8.2] K. Knopp: Theorie und Anwendung der uendlichen Reihen. Springer, 5. Aufl. 1964.