Potential of homogeneously charged ellipse

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Kapchinsky, Theory of resonance linear accelerators. Harwood, 1992. pp.255/6 .

Expansions of inner and outer potential

ρ = constant charge density; e = excentricity, $c = -\rho\, e^2/(8\, \varepsilon_0)$.

np = number of multipoles used

```
np = 5;
```

```
st = Table[0, {np + 1}];
```

```
Ui = c (Cosh[2 η] + Cos[2 ψ]) + Sum[a[2 n] Cosh[2 n η] Cos[2 n ψ], {n, 0, np}]
```

```
a[0] + a[2] Cos[2 ψ] Cosh[2 η] + c (Cos[2 ψ] + Cosh[2 η]) + a[4] Cos[4 ψ] Cosh[4 η] +
 a[6] Cos[6 ψ] Cosh[6 η] + a[8] Cos[8 ψ] Cosh[8 η] + a[10] Cos[10 ψ] Cosh[10 η]
```

```
Ue = b[0] η + Sum[b[2 n] Exp[- 2 n η] Cos[2 n ψ], {n, np}]
```

$\eta\, b[0] + e^{-2\eta}\, b[2]\, Cos[2\psi] + e^{-4\eta}\, b[4]\, Cos[4\psi] +$
$e^{-6\eta}\, b[6]\, Cos[6\psi] + e^{-8\eta}\, b[8]\, Cos[8\psi] + e^{-10\eta}\, b[10]\, Cos[10\psi]$

```
dUi = D[Ui, η]
```

```
2 c Sinh[2 η] + 2 a[2] Cos[2 ψ] Sinh[2 η] + 4 a[4] Cos[4 ψ] Sinh[4 η] +
 6 a[6] Cos[6 ψ] Sinh[6 η] + 8 a[8] Cos[8 ψ] Sinh[8 η] + 10 a[10] Cos[10 ψ] Sinh[10 η]
```

```
dUe = D[Ue, η]
```

$b[0] - 2\, e^{-2\eta}\, b[2]\, Cos[2\psi] - 4\, e^{-4\eta}\, b[4]\, Cos[4\psi] -$
$6\, e^{-6\eta}\, b[6]\, Cos[6\psi] - 8\, e^{-8\eta}\, b[8]\, Cos[8\psi] - 10\, e^{-10\eta}\, b[10]\, Cos[10\psi]$

```
se = η → η0;
```

n = 0

```
n = 0;
```

```
su = Thread[Table[Cos[2 n ψ] → 0, {n, np}]]
```

```
{Cos[2 ψ] → 0, Cos[4 ψ] → 0, Cos[6 ψ] → 0, Cos[8 ψ] → 0, Cos[10 ψ] → 0}
```

```
eq1 = (Ui /. su /. se) == (Ue /. su /. se)
```

a[0] + c Cosh[2 η0] == η0 b[0]

```
eq2 = (dUi /. su /. se) == (dUe /. su /. se)
```

2 c Sinh[2 η0] == b[0]

```
st[[n + 1]] = Solve[{eq1, eq2}, {a[n], b[n]}]
```

{{a[0] → -c Cosh[2 η0] + 2 c η0 Sinh[2 η0], b[0] → 2 c Sinh[2 η0]}}

■ n = 1

```
n = 1;
eq1 = Coefficient[Ui /. se, Cos[2 ψ]] == Coefficient[Ue /. se, Cos[2 ψ]]
```

$c + a[2] \, \mathrm{Cosh}[2\,\eta0] == e^{-2\,\eta0}\, b[2]$

```
eq2 = Coefficient[dUi /. se, Cos[2 ψ]] == Coefficient[dUe /. se, Cos[2 ψ]]
```

$2\,a[2] \, \mathrm{Sinh}[2\,\eta0] == -2\, e^{-2\,\eta0}\, b[2]$

```
s1 = Solve[{eq1, eq2}, {a[2], b[2]}] // Flatten
```

$\{a[2] \to -c\,\mathrm{Cosh}[2\,\eta0] + c\,\mathrm{Sinh}[2\,\eta0],\ b[2] \to \frac{1}{2}\, c\, e^{-2\,\eta0}\, (-1 + e^{4\,\eta0})\}$

```
Map[TrigToExp, s1, {2}]
```

$\{a[2] \to -c\, e^{-2\,\eta0},\ b[2] \to -\frac{1}{2}\, c\, e^{-2\,\eta0} + \frac{1}{2}\, c\, e^{2\,\eta0}\}$

```
st[[n + 1]] = %
```

$\{a[2] \to -c\, e^{-2\,\eta0},\ b[2] \to -\frac{1}{2}\, c\, e^{-2\,\eta0} + \frac{1}{2}\, c\, e^{2\,\eta0}\}$

■ n = 2

```
n = 2;
eq1 = Coefficient[Ui /. se, Cos[2 n ψ]] == Coefficient[Ue /. se, Cos[2 n ψ]]
```

$a[4] \, \mathrm{Cosh}[4\,\eta0] == e^{-4\,\eta0}\, b[4]$

```
eq2 = Coefficient[dUi /. se, Cos[2 n ψ]] == Coefficient[dUe /. se, Cos[2 n ψ]]
```

$4\,a[4] \, \mathrm{Sinh}[4\,\eta0] == -4\, e^{-4\,\eta0}\, b[4]$

```
st[[n + 1]] = Solve[{eq1, eq2}, {a[2 n], b[2 n]}] // Flatten
```

{a[4] → 0, b[4] → 0}

■ n = 3

```
n = 3;
```

```
eq1 = Coefficient[Ui /. se, Cos[2 n ψ]] == Coefficient[Ue /. se, Cos[2 n ψ]]
```

$a[6]\ \mathrm{Cosh}[6\,\eta 0] == e^{-6\,\eta 0}\ b[6]$

```
eq2 = Coefficient[dUi /. se, Cos[2 n ψ]] == Coefficient[dUe /. se, Cos[2 n ψ]]
```

$6\ a[6]\ \mathrm{Sinh}[6\,\eta 0] == -6\ e^{-6\,\eta 0}\ b[6]$

```
st[[n + 1]]  = Solve[{eq1, eq2}, {a[2 n], b[2 n]}] // Flatten
```

{a[6] → 0, b[6] → 0}

▪ n = 4

```
n = 4;
```

```
eq1 = Coefficient[Ui /. se, Cos[2 n ψ]] == Coefficient[Ue /. se, Cos[2 n ψ]]
```

$a[8]\ \mathrm{Cosh}[8\,\eta 0] == e^{-8\,\eta 0}\ b[8]$

```
eq2 = Coefficient[dUi /. se, Cos[2 n ψ]] == Coefficient[dUe /. se, Cos[2 n ψ]]
```

$8\ a[8]\ \mathrm{Sinh}[8\,\eta 0] == -8\ e^{-8\,\eta 0}\ b[8]$

```
st[[n + 1]]  = Solve[{eq1, eq2}, {a[2 n], b[2 n]}] // Flatten
```

{a[8] → 0, b[8] → 0}

▪ n = 5

```
n = 5;
```

```
eq1 = Coefficient[Ui /. se, Cos[2 n ψ]] == Coefficient[Ue /. se, Cos[2 n ψ]]
```

$a[10]\ \mathrm{Cosh}[10\,\eta 0] == e^{-10\,\eta 0}\ b[10]$

```
eq2 = Coefficient[dUi /. se, Cos[2 n ψ]] == Coefficient[dUe /. se, Cos[2 n ψ]]
```

$10\ a[10]\ \mathrm{Sinh}[10\,\eta 0] == -10\ e^{-10\,\eta 0}\ b[10]$

```
st[[n + 1]] = Solve[{eq1, eq2}, {a[2 n], b[2 n]}] // Flatten
```

{a[10] → 0, b[10] → 0}

▪ Result

```
st
```

$$\{\{\{a[0] \to -c\,\mathrm{Cosh}[2\,\eta 0] + 2\,c\,\eta 0\,\mathrm{Sinh}[2\,\eta 0],\ b[0] \to 2\,c\,\mathrm{Sinh}[2\,\eta 0]\}\},$$
$$\{a[2] \to -c\,e^{-2\,\eta 0},\ b[2] \to -\frac{1}{2}\,c\,e^{-2\,\eta 0} + \frac{1}{2}\,c\,e^{2\,\eta 0}\},\ \{a[4] \to 0,\ b[4] \to 0\},$$
$$\{a[6] \to 0,\ b[6] \to 0\},\ \{a[8] \to 0,\ b[8] \to 0\},\ \{a[10] \to 0,\ b[10] \to 0\}\}$$

```
Uint = Ui /. Flatten[st]
```

$$-c\,e^{-2\,\eta 0}\,\mathrm{Cos}[2\,\psi]\,\mathrm{Cosh}[2\,\eta] + c\,(\mathrm{Cos}[2\,\psi] + \mathrm{Cosh}[2\,\eta]) - c\,\mathrm{Cosh}[2\,\eta 0] + 2\,c\,\eta 0\,\mathrm{Sinh}[2\,\eta 0]$$

```
c1 = Uint /. {Cos[2 ψ] → 0, Cosh[2 η] → 0}
```

$$\frac{e^2\,\rho 0\,\mathrm{Cosh}[2\,\eta 0]}{8\,\varepsilon 0}-\frac{e^2\,\eta 0\,\rho 0\,\mathrm{Sinh}[2\,\eta 0]}{4\,\varepsilon 0}$$

```
Ue /. Flatten[st]
```

$$e^{-2\,\eta}\left(-\frac{1}{2}\,c\,e^{-2\,\eta 0}+\frac{1}{2}\,c\,e^{2\,\eta 0}\right)\mathrm{Cos}[2\,\psi]+2\,c\,\eta\,\mathrm{Sinh}[2\,\eta 0]$$

$$\left(-\frac{1}{2}\,c\,e^{-2\,\eta 0}+\frac{1}{2}\,c\,e^{2\,\eta 0}\right)\ \text{// ExpToTrig}$$

$$c\,\mathrm{Sinh}[2\,\eta 0]$$

■ Returning to Cartesian coordinates

```
x = e Cosh[η] Cos[ψ]
 y = e Sinh[η] Sin[ψ]
```

$$e\,\mathrm{Cos}[\psi]\,\mathrm{Cosh}[\eta]$$

$$e\,\mathrm{Sin}[\psi]\,\mathrm{Sinh}[\eta]$$

In particular for the semi-axes we have:

```
TrigReduce[x^2 + y^2] // Factor
```

$$\frac{1}{2}\,e^2\,(\mathrm{Cos}[2\,\psi]+\mathrm{Cosh}[2\,\eta])$$

```
TrigReduce[x^2 - y^2] // ComplexExpand // Factor
```

$$\frac{1}{2}\,e^2\,(1+\mathrm{Cos}[2\,\psi]\,\mathrm{Cosh}[2\,\eta])$$

```
a = e Cosh[η0]
```

$$e\,\mathrm{Cosh}[\eta 0]$$

```
b = e Sinh[η0]
```

$$e\,\mathrm{Sinh}[\eta 0]$$

```
(a - b) / (a + b) // TrigToExp // Simplify
```

$$e^{-2\,\eta 0}$$

```
Clear[a, b, x, y]
```

```
suxpy = {Cos[2 ψ] + Cosh[2 η] → 2 (x^2 + y^2) / e^2}
```

$$\left\{\mathrm{Cos}[2\,\psi]+\mathrm{Cosh}[2\,\eta]\to\frac{2\,(x^2+y^2)}{e^2}\right\}$$

```
suxmy = {Cos[2 ψ] Cosh[2 η] → ( 2 (x^2 - y^2) / e^2 - 1)}
```

$$\left\{\mathrm{Cos}[2\,\psi]\,\mathrm{Cosh}[2\,\eta]\to -1+\frac{2\,(x^2-y^2)}{e^2}\right\}$$

```
suab = {ⅇ^(-2 η0) → (a - b) / (a + b)}
```

$$\left\{e^{-2\,\eta 0} \to \frac{a-b}{a+b}\right\}$$

```
c = - ρ0 e^2 / 8 / ε0
```

$$-\frac{e^2\,\rho 0}{8\,\varepsilon 0}$$

```
Uint /. suab
```

$$\frac{(a-b)\,e^2\,\rho 0\,\text{Cos}[2\,\psi]\,\text{Cosh}[2\,\eta]}{8\,(a+b)\,\varepsilon 0} - \frac{e^2\,\rho 0\,(\text{Cos}[2\,\psi]+\text{Cosh}[2\,\eta])}{8\,\varepsilon 0} + \frac{e^2\,\rho 0\,\text{Cosh}[2\,\eta 0]}{8\,\varepsilon 0} - \frac{e^2\,\eta 0\,\rho 0\,\text{Sinh}[2\,\eta 0]}{4\,\varepsilon 0}$$

```
Uxyi = % /. Join[suxmy, suxpy] // Cancel
```

$$-\frac{(x^2+y^2)\,\rho 0}{4\,\varepsilon 0} - \frac{(a-b)\,(e^2-2\,x^2+2\,y^2)\,\rho 0}{8\,(a+b)\,\varepsilon 0} + \frac{e^2\,\rho 0\,\text{Cosh}[2\,\eta 0]}{8\,\varepsilon 0} - \frac{e^2\,\eta 0\,\rho 0\,\text{Sinh}[2\,\eta 0]}{4\,\varepsilon 0}$$

```
Uxyi - c1
```

$$-\frac{(x^2+y^2)\,\rho 0}{4\,\varepsilon 0} - \frac{(a-b)\,(e^2-2\,x^2+2\,y^2)\,\rho 0}{8\,(a+b)\,\varepsilon 0}$$

```
c1
```

$$\frac{e^2\,\rho 0\,\text{Cosh}[2\,\eta 0]}{8\,\varepsilon 0} - \frac{e^2\,\eta 0\,\rho 0\,\text{Sinh}[2\,\eta 0]}{4\,\varepsilon 0}$$