26.5, Satz 1: Hill's equation: Even and odd solutions.

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Hill's equation $d^2 w \big/ \text{dz}^2 + \text{J(z) w(z0)} = 0$
has solutions, which may be chosen such that one is even, which is denoted as **c(z)**,
while the other one is odd in z and is denoted as **s(z).**
The proof below uses power series expansions of the solutions (Frobenius method).

Power series of the solution and its second derivative

```
sw = Sum[a[k] z^(k + ρ) , {k, 0, 4}]
```

$$z^{\rho}\, a[0] + z^{1+\rho}\, a[1] + z^{2+\rho}\, a[2] + z^{3+\rho}\, a[3] + z^{4+\rho}\, a[4]$$

```
t1 = D[sw, {z, 2}]
```

$$z^{-2+\rho} (-1+\rho)\, \rho\, a[0] + z^{-1+\rho}\, \rho\, (1+\rho)\, a[1] + z^{\rho} (1+\rho)(2+\rho)\, a[2] + z^{1+\rho} (2+\rho)(3+\rho)\, a[3] + z^{2+\rho} (3+\rho)(4+\rho)\, a[4]$$

Power series of J(z) and of the product J(z) w(z)

```
sjs = Series[j[z], {z, 0, 4}]
```

$$j[0] + j'[0]\, z + \frac{1}{2} j''[0]\, z^2 + \frac{1}{6} j^{(3)}[0]\, z^3 + \frac{1}{24} j^{(4)}[0]\, z^4 + O[z]^5$$

```
t2 = Expand[sw sjs]
```

$$z^{\rho} \left(a[0]\, j[0] + a[0]\, j'[0]\, z + \frac{1}{2} a[0]\, j''[0]\, z^2 + \frac{1}{6} a[0]\, j^{(3)}[0]\, z^3 + \frac{1}{24} a[0]\, j^{(4)}[0]\, z^4 + O[z]^5\right) +$$
$$z^{1+\rho} \left(a[1]\, j[0] + a[1]\, j'[0]\, z + \frac{1}{2} a[1]\, j''[0]\, z^2 + \frac{1}{6} a[1]\, j^{(3)}[0]\, z^3 + \frac{1}{24} a[1]\, j^{(4)}[0]\, z^4 + O[z]^5\right) +$$
$$z^{2+\rho} \left(a[2]\, j[0] + a[2]\, j'[0]\, z + \frac{1}{2} a[2]\, j''[0]\, z^2 + \frac{1}{6} a[2]\, j^{(3)}[0]\, z^3 + \frac{1}{24} a[2]\, j^{(4)}[0]\, z^4 + O[z]^5\right) +$$
$$z^{3+\rho} \left(a[3]\, j[0] + a[3]\, j'[0]\, z + \frac{1}{2} a[3]\, j''[0]\, z^2 + \frac{1}{6} a[3]\, j^{(3)}[0]\, z^3 + \frac{1}{24} a[3]\, j^{(4)}[0]\, z^4 + O[z]^5\right) +$$
$$z^{4+\rho} \left(a[4]\, j[0] + a[4]\, j'[0]\, z + \frac{1}{2} a[4]\, j''[0]\, z^2 + \frac{1}{6} a[4]\, j^{(3)}[0]\, z^3 + \frac{1}{24} a[4]\, j^{(4)}[0]\, z^4 + O[z]^5\right)$$

- **The resulting ower series of the solution**

```
tt = t1 + t2 // Normal // Expand
```

$$-z^{-2+\rho}\rho\,a[0]+z^{-2+\rho}\rho^2\,a[0]+z^{-1+\rho}\rho\,a[1]+z^{-1+\rho}\rho^2\,a[1]+2\,z^{\rho}\,a[2]+3\,z^{\rho}\rho\,a[2]+z^{\rho}\rho^2\,a[2]+6\,z^{1+\rho}\,a[3]+5\,z^{1+\rho}\rho\,a[3]+z^{1+\rho}\rho^2\,a[3]+12\,z^{2+\rho}\,a[4]+7\,z^{2+\rho}\rho\,a[4]+z^{2+\rho}\rho^2\,a[4]+z^{\rho}\,a[0]\,j[0]+z^{1+\rho}\,a[1]\,j[0]+z^{2+\rho}\,a[2]\,j[0]+z^{3+\rho}\,a[3]\,j[0]+z^{4+\rho}\,a[4]\,j[0]+z^{1+\rho}\,a[0]\,j'[0]+z^{2+\rho}\,a[1]\,j'[0]+z^{3+\rho}\,a[2]\,j'[0]+z^{4+\rho}\,a[3]\,j'[0]+z^{5+\rho}\,a[4]\,j'[0]+\frac{1}{2}z^{2+\rho}\,a[0]\,j''[0]+\frac{1}{2}z^{3+\rho}\,a[1]\,j''[0]+\frac{1}{2}z^{4+\rho}\,a[2]\,j''[0]+\frac{1}{2}z^{5+\rho}\,a[3]\,j''[0]+\frac{1}{2}z^{6+\rho}\,a[4]\,j''[0]+\frac{1}{6}z^{3+\rho}\,a[0]\,j^{(3)}[0]+\frac{1}{6}z^{4+\rho}\,a[1]\,j^{(3)}[0]+\frac{1}{6}z^{5+\rho}\,a[2]\,j^{(3)}[0]+\frac{1}{6}z^{6+\rho}\,a[3]\,j^{(3)}[0]+\frac{1}{6}z^{7+\rho}\,a[4]\,j^{(3)}[0]+\frac{1}{24}z^{4+\rho}\,a[0]\,j^{(4)}[0]+\frac{1}{24}z^{5+\rho}\,a[1]\,j^{(4)}[0]+\frac{1}{24}z^{6+\rho}\,a[2]\,j^{(4)}[0]+\frac{1}{24}z^{7+\rho}\,a[3]\,j^{(4)}[0]+\frac{1}{24}z^{8+\rho}\,a[4]\,j^{(4)}[0]$$

- **The first two terms of the power series**

```
h1  = Coefficient[tt, z^(-2+ρ)] z^(-2+ρ) // Factor
```

$$z^{-2+\rho}\,(-1+\rho)\,\rho\,a[0]$$

```
h2  = Coefficient[tt, z^(-1+ρ)] z^(-1+ρ) // Factor
```

$$z^{-1+\rho}\,\rho\,(1+\rho)\,a[1]$$

- **The remaining terms of the power series**

```
tr = Drop[tt, 4] / z^ρ // Expand
```

$$2\,a[2]+3\,\rho\,a[2]+\rho^2\,a[2]+6\,z\,a[3]+5\,z\,\rho\,a[3]+z\,\rho^2\,a[3]+12\,z^2\,a[4]+7\,z^2\rho\,a[4]+z^2\rho^2\,a[4]+a[0]\,j[0]+z\,a[1]\,j[0]+z^2\,a[2]\,j[0]+z^3\,a[3]\,j[0]+z^4\,a[4]\,j[0]+z\,a[0]\,j'[0]+z^2\,a[1]\,j'[0]+z^3\,a[2]\,j'[0]+z^4\,a[3]\,j'[0]+z^5\,a[4]\,j'[0]+\frac{1}{2}z^2\,a[0]\,j''[0]+\frac{1}{2}z^3\,a[1]\,j''[0]+\frac{1}{2}z^4\,a[2]\,j''[0]+\frac{1}{2}z^5\,a[3]\,j''[0]+\frac{1}{2}z^6\,a[4]\,j''[0]+\frac{1}{6}z^3\,a[0]\,j^{(3)}[0]+\frac{1}{6}z^4\,a[1]\,j^{(3)}[0]+\frac{1}{6}z^5\,a[2]\,j^{(3)}[0]+\frac{1}{6}z^6\,a[3]\,j^{(3)}[0]+\frac{1}{6}z^7\,a[4]\,j^{(3)}[0]+\frac{1}{24}z^4\,a[0]\,j^{(4)}[0]+\frac{1}{24}z^5\,a[1]\,j^{(4)}[0]+\frac{1}{24}z^6\,a[2]\,j^{(4)}[0]+\frac{1}{24}z^7\,a[3]\,j^{(4)}[0]+\frac{1}{24}z^8\,a[4]\,j^{(4)}[0]$$

- **J(z) is an even function in z**

```
tr = tr /. {j'[0] → 0, j^(3)[0] → 0}
```

$$2\,a[2]+3\,\rho\,a[2]+\rho^2\,a[2]+6\,z\,a[3]+5\,z\,\rho\,a[3]+z\,\rho^2\,a[3]+12\,z^2\,a[4]+7\,z^2\rho\,a[4]+z^2\rho^2\,a[4]+a[0]\,j[0]+z\,a[1]\,j[0]+z^2\,a[2]\,j[0]+z^3\,a[3]\,j[0]+z^4\,a[4]\,j[0]+\frac{1}{2}z^2\,a[0]\,j''[0]+\frac{1}{2}z^3\,a[1]\,j''[0]+\frac{1}{2}z^4\,a[2]\,j''[0]+\frac{1}{2}z^5\,a[3]\,j''[0]+\frac{1}{2}z^6\,a[4]\,j''[0]+\frac{1}{24}z^4\,a[0]\,j^{(4)}[0]+\frac{1}{24}z^5\,a[1]\,j^{(4)}[0]+\frac{1}{24}z^6\,a[2]\,j^{(4)}[0]+\frac{1}{24}z^7\,a[3]\,j^{(4)}[0]+\frac{1}{24}z^8\,a[4]\,j^{(4)}[0]$$

The first term of the remaining power series

```
tr /. z → 0
```

$2\,a[2] + 3\,\rho\,a[2] + \rho^2\,a[2] + a[0]\,j[0]$

```
h3 = % z^ρ
```

$z^{\rho}\,\left(2\,a[2] + 3\,\rho\,a[2] + \rho^2\,a[2] + a[0]\,j[0]\right)$

The second term of the remaining power series

```
D[tr, z] /. z → 0
```

$6\,a[3] + 5\,\rho\,a[3] + \rho^2\,a[3] + a[1]\,j[0]$

```
h4 = % z^(ρ + 1)
```

$z^{1+\rho}\,\left(6\,a[3] + 5\,\rho\,a[3] + \rho^2\,a[3] + a[1]\,j[0]\right)$

The third term of the remaining power series

```
D[tr, {z, 2}] /. z → 0
```

$24\,a[4] + 14\,\rho\,a[4] + 2\,\rho^2\,a[4] + 2\,a[2]\,j[0] + a[0]\,j''[0]$

```
h5 = % z^(ρ+2)
```

$z^{2+\rho}\,\left(24\,a[4] + 14\,\rho\,a[4] + 2\,\rho^2\,a[4] + 2\,a[2]\,j[0] + a[0]\,j''[0]\right)$

The first few terms of the total power series

```
hi = h1 + h2 + h3 + h4 + h5
```

$$z^{-2+\rho}\,(-1+\rho)\,\rho\,a[0] + z^{-1+\rho}\,\rho\,(1+\rho)\,a[1] + z^{\rho}\,\left(2\,a[2] + 3\,\rho\,a[2] + \rho^2\,a[2] + a[0]\,j[0]\right) + z^{1+\rho}\,\left(6\,a[3] + 5\,\rho\,a[3] + \rho^2\,a[3] + a[1]\,j[0]\right) + z^{2+\rho}\,\left(24\,a[4] + 14\,\rho\,a[4] + 2\,\rho^2\,a[4] + 2\,a[2]\,j[0] + a[0]\,j''[0]\right)$$

The first two terms must be zero. This determins the values of ρ as 0 or 1,
in the second case also the value of a[1].

ρ = 0, a[0] and a[1] arbitrary, all other a[n] are determined by recurrences

```
h1
h2
```

$z^{-2+\rho}\,(-1+\rho)\,\rho\,a[0]$

$z^{-1+\rho}\,\rho\,(1+\rho)\,a[1]$

```
h1 /. ρ → 0
h2 /. ρ → 0
```

0

0

- **The first few terms of the remaining part of the resulting power series**

```
hi1 = hi /. ρ → 0
```

$2\,a[2] + a[0]\,j[0] + z\,(6\,a[3] + a[1]\,j[0]) + z^2\,(24\,a[4] + 2\,a[2]\,j[0] + a[0]\,j''[0])$

```
eq10 = (hi1 /. z → 0) == 0
```

$2\,a[2] + a[0]\,j[0] == 0$

```
eq11  = Coefficient[hi1, z] == 0
```

$6\,a[3] + a[1]\,j[0] == 0$

```
eq12 = Coefficient[hi1, z^2] == 0
```

$24\,a[4] + 2\,a[2]\,j[0] + a[0]\,j''[0] == 0$

```
so1 = Solve[{eq10, eq11, eq12}, {a[2], a[3], a[4]}] // Flatten
```

$\left\{a[4] \to \frac{1}{24}\left(a[0]\,j[0]^2 - a[0]\,j''[0]\right),\ a[3] \to -\frac{1}{6}\,a[1]\,j[0],\ a[2] \to -\frac{1}{2}\,a[0]\,j[0]\right\}$

- **a[1] is chosen as zero. The resulting shape of the final form of the first solution, which is an even function :**

```
s1 = sw /. Join[{ρ → 0}, so1]  /. a[1] → 0
```

$a[0] - \frac{1}{2}\,z^2\,a[0]\,j[0] + \frac{1}{24}\,z^4\left(a[0]\,j[0]^2 - a[0]\,j''[0]\right)$

- **ρ = 1, a[0] = b[0] and a[1] arbitrary, a[1] = 0** chosen; **all other a[n] are determined by recurrences**

```
h1
h2
```

$z^{-2+\rho}\,(-1+\rho)\,\rho\,a[0]$

$z^{-1+\rho}\,\rho\,(1+\rho)\,a[1]$

```
h1 /. ρ → 1
h2 /. {ρ → 1, a[1] → 0}
```

0

0

- **The first few terms of the remaining part of the resulting power series**

```
hi2 = hi /. ρ → 1
```

$2\,a[1] + z\,(6\,a[2] + a[0]\,j[0]) + z^2\,(12\,a[3] + a[1]\,j[0]) + z^3\,(40\,a[4] + 2\,a[2]\,j[0] + a[0]\,j''[0])$

```
eq20 = (hi2 /. z → 0) == 0
```

$2\,a[1] == 0$

```
eq21 = Coefficient[hi2, z] == 0
```

$6\,a[2] + a[0]\,j[0] == 0$

```
eq22 = Coefficient[hi2, z^2] == 0
```

$12\,a[3] + a[1]\,j[0] == 0$

```
so2 = Solve[{eq20, eq21, eq22}, {a[2], a[3], a[4]}] // Flatten
```

$\left\{a[2] \to -\frac{1}{6} a[0]\, j[0],\ a[3] \to 0\right\}$

- **a[1] is chosen as zero. The resulting shape of the final form of the first solution, which is an odd function :**

```
s2 = sw /. Join[{ρ → 1, a[1] → 0}, so2] /. a[ii_] → b[ii]
```

$z\, b[0] + z^5\, b[4] - \frac{1}{6} z^3\, b[0]\, j[0]$