Chapter 25

Green's Tensors for the Electromagnetic Field in Cylindrical Coordinates

The solutions of the vector Helmholtz equation in three dimensions can be expressed by a complete set of vector fields denoted as $\mathbf{L}, \mathbf{M}, \mathbf{N}$. $\mathbf{L}$ is a source field, the other two fields are solenoidal. This set can also be used for Maxwells equations. However, in source-free regions of space the two solenoidal fields suffice to express all solutions. Green's tensors of the electromagentic fields were derived with fields of all three types. But in the last section of the preceeding chapter it was shown that the source fields may be removed in such a way that only the solenoidal fields plus a singular term remain. In this chapter the Green's tensor of free space is derived in cylindrical coordinates r, θ, ϕ. It is shown that this removal of the source field can be done in two ways leading to two differenent representations of the Greens tensor; the singular term is then proportional either to the dyadic $\mathbf{e}_z\mathbf{e}_{z'}$ or $\mathbf{e}_r\mathbf{e}_{r'}$.

25.1 The vector fields, Hansen Harmonics

The three types of vector fields solving the vector Helmholtz equation in cylindrical coordinates r, θ, ϕ can be derived from the following set of scalar functions:

$$\psi_{mh\lambda}(r, \theta, \phi) = J_m(\lambda r)\ e^{im\phi}\ e^{ih\lambda z}, \quad m \in \mathbb{Z},\ 0 \leq \lambda < \infty,\ -\infty < h < \infty, \tag{25.1}$$

which are particular solutions of the scalar Helmholtz equation $\Delta\psi + k^2\psi = 0$ provided

$$k^2 = h^2 + \lambda^2. \tag{25.2}$$

These functions form a compete set of solutions if the parameters m, h, λ are allowed to vary through the full range indicated in the definition (25.1). For an integral representation of a solution the paths of integration in the complex $\lambda-$ and $h-$planes must be suitably chosen to avoid a vanishing of the denominators of the integrands and to ensure the Sommerfeld radiation condition. This will be treated in more detail below. The time-dependence $e^{-i\omega t}$ is suppressed throughout this chapter.

The solenoidal fields are defined as

$$\mathbf{M}_{mh\lambda}(r, \theta, \phi) = \nabla \times \mathbf{e}_z \psi_{mh\lambda}(r, \theta, \phi) := \mathbf{T}^M \psi_{mh\lambda}(r, \theta, \phi) \tag{25.3}$$

with the operator

$$\mathbf{T}^M = \mathbf{e}_r \frac{1}{r}\partial_\phi - \mathbf{e}_\phi \partial_r; \tag{25.4}$$

and

$$\mathbf{N}_{mh\lambda}(r, \theta, \phi) = \frac{1}{\sqrt{\lambda^2 + h^2}} \nabla \times\ \mathbf{M}_{mh\lambda}(r, \theta, \phi) := \frac{1}{\sqrt{\lambda^2 + h^2}} \mathbf{T}^N \psi_{mh\lambda}(r, \theta, \phi) \tag{25.5}$$

with the operator

$$\mathbf{T}^N = \mathbf{e}_r \frac{1}{r} \partial_{zr} + \mathbf{e}_\phi \partial_{z\ phi} - \mathbf{e}_z \Big(\partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\phi\phi} \Big). \tag{25.6}$$

Note that the coefficient of $\mathbf{e}_z$ becomes λ^2 if the operator $\mathbf{T}^N$ acts on $\psi_{mh\lambda}(r, \theta, \phi)$.

The irrotational fields are :

$$\mathbf{L}_{mh\lambda}(r, \theta, \phi) = \nabla \psi_{mh\lambda}(r, \theta, \phi) := \mathbf{T}^L \psi_{mh\lambda}(r, \theta, \phi) \tag{25.7}$$

with the operator $\mathbf{L}$ being equal to the nabla operator in circular cylindrical coordinates.

The vector fields fulfil the following orthogonality relations:

$$\begin{aligned} &\int_{-\infty}^{\infty} dz \int_0^{\infty} r\ dr \int_0^{2\pi} d\phi\ \big(\mathbf{M}_{mh\lambda}(r, \theta, \phi) \cdot \mathbf{M}^*_{m'h'\lambda'}(r, \theta, \phi) \big) = \\ &\int_{-\infty}^{\infty} dz \int_0^{\infty} r\ dr \int_0^{2\pi} d\phi\ \big(\mathbf{N}_{mh\lambda}(r, \theta, \phi) \cdot \mathbf{N}^*_{m'h'\lambda'}(r, \theta, \phi) \big) = \\ &= 4\pi^2\ \delta_{mm'}\ \delta(h-h')\ \delta(\lambda - \lambda')\ \lambda \end{aligned} \tag{25.8}$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty} dz \int_0^{\infty} r\ dr \int_0^{2\pi} d\phi\ \big(\mathbf{L}_{mh\lambda}(r, \theta, \phi) \cdot \mathbf{L}^*_{m'h'\lambda'}(r, \theta, \phi) \big) = \\ &= 4\pi^2\ \delta_{mm'}\ \delta(h-h')\ \delta(\lambda - \lambda')\ (h^2 + \lambda^2)/\lambda. \end{aligned} \tag{25.9}$$

all integrals containing mixed scalar products as $(\mathbf{L} \cdot \mathbf{M}), (\mathbf{L} \cdot \mathbf{N}), (\mathbf{M} \cdot \mathbf{N})$, are zero.

25.2 Green's tensors

Green's tensors for the electromagnetic field are defined as solutions of the following inhomogeneous equation:

$$\nabla \times (\nabla \times \mathbf{\Gamma}(\mathbf{r}; \mathbf{r}')) \quad - \quad k^2\ \mathbf{\Gamma}(\mathbf{r}; \mathbf{r}') \ = \ \mathbf{I}\ \delta(\mathbf{r} - \mathbf{r}'). \tag{25.10}$$

Here only the tensor for free space is considered. Therefore superscripts are omitted. The wave number $k = \omega / c$ has the usual meaning. At infinity the tensor must fulfil the radiation condition.

The electric field excited by an electric current distribution $\mathbf{Jr}$ is given by:

$$\mathbf{E}(\mathbf{r}) \ = \ i\omega\mu_0 \int\int\int\ \mathbf{\Gamma}(\mathbf{r}; \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\ d\mathbf{r}'; \tag{25.11}$$

the magnetic field due to a magnetic current distribution $\mathbf{Mr}$ may be expressed with the help of the same tensor:

$$\mathbf{H}(\mathbf{r}) \ = \ i\omega\varepsilon_0 \int\int\int\ \mathbf{\Gamma}(\mathbf{r}; \mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')\ d\mathbf{r}'. \tag{25.12}$$

In cylindrical coordinates the tensor is given by the following matrix:

$$\mathbf{\Gamma}(\mathbf{r}, \mathbf{r}') \ = \ \begin{pmatrix} \Gamma_{rr}(r, \theta, \phi; r', \theta', \phi') & \Gamma_{r\phi}(r, \theta, \phi; r', \theta', \phi') & \Gamma_{rz}(r, \theta, \phi; r', \theta', \phi') \\ \Gamma_{\phi r}(r, \theta, \phi; r', \theta', \phi') & \Gamma_{\phi\phi}(r, \theta, \phi; r', \theta', \phi') & \Gamma_{\phi z}(r, \theta, \phi; r', \theta', \phi') \\ \Gamma_{zr}(r, \theta, \phi; r', \theta', \phi') & \Gamma_{z\phi}(r, \theta, \phi; r', \theta', \phi') & \Gamma_{zz}(r, \theta, \phi; r', \theta', \phi') \end{pmatrix} \tag{25.13}$$

and the electric current density by:

$$\mathbf{J}(\mathbf{r}) \ = \ (J_r(r, \theta, \phi), J_\phi(r, \theta, \phi), J_z(r, \theta, \phi)). \tag{25.14}$$

25.2.1 General integral representation of the Green's tensor in Hansen harmonics

The completeness relation of the vector fields defined in the second section is:

$$\mathbf{I}\;\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_{m=-\infty}^{\infty} \left[\frac{\mathbf{M}_{mh\lambda}(\mathbf{r})\mathbf{M}^*_{mh\lambda}(\mathbf{r}') + \mathbf{N}_{mh\lambda}(\mathbf{r})\mathbf{N}^*_{mh\lambda}(\mathbf{r}')}{\lambda} + \right.$$
$$\left. + \frac{\lambda}{\lambda^2+h^2}\,\mathbf{L}_{mh\lambda}(\mathbf{r})\mathbf{L}^*_{mh\lambda}(\mathbf{r}')\right]. \qquad (25.15)$$

A similar expansion with unknown coefficients is set up for the Green's tensor. This and the above completeness relation are inserted into (25.10) and coefficients of like terms are compared. This gives the following representation of the Green's tensor:

$$\mathbf{\Gamma}(\mathbf{r};\mathbf{r}') \;=\; \frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\mathbf{\Gamma}_m(\mathbf{r};\mathbf{r}') \qquad (25.16)$$

with

$$\mathbf{\Gamma}_m(\mathbf{r};\mathbf{r}') \;=\; \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_{m=-\infty}^{\infty} \left[\frac{\mathbf{M}_{mh\lambda}(\mathbf{r})\mathbf{M}^*_{mh\lambda}(\mathbf{r}') + \mathbf{N}_{mh\lambda}(\mathbf{r})\mathbf{N}^*_{mh\lambda}(\mathbf{r}')}{\lambda(\lambda^2+h^2-k^2)} - \right.$$
$$\left. - \frac{\lambda}{k^2(\lambda^2+h^2)}\,\mathbf{L}_{mh\lambda}(\mathbf{r})\mathbf{L}^*_{mh\lambda}(\mathbf{r}')\right] \qquad (25.17)$$
$$= \lim_{\varepsilon\to 0}\left(\mathbf{T}^M\mathbf{T}^{M'}M_m^{(-1)} + \mathbf{T}^N\mathbf{T}^{N'}N_m^{(-1)} - \mathbf{T}^L\mathbf{T}^{L'}L_m^{(-1)}\right)\; e^{im(\phi-\phi')}. \qquad (25.18)$$

The integration paths in the h- and $\lambda-$ planes must avoid the points where the denominators become zero; this must be done such that the radiation condition is fulfilled. This is worked out below (conf. Figs.1 and 2). The integrals occuring in these expressions are defined as:

$$M_m^{(\nu)} \;=\; \int_{-\infty}^{\infty} dh \int_{\varepsilon}^{\infty} d\lambda\; \lambda^\nu\; \frac{J_m(\lambda r)J_m(\lambda r')\;e^{ih(z-z')}}{(\lambda^2+h^2-k^2)}, \qquad (25.19)$$

$$N_m^{(\nu)} \;=\; \int_{-\infty}^{\infty} dh \int_{\varepsilon}^{\infty} d\lambda\; \lambda^\nu\; \frac{J_m(\lambda r)J_m(\lambda r')\;e^{ih(z-z')}}{(\lambda^2+h^2)(\lambda^2+h^2-k^2)}, \qquad (25.20)$$

$$L_m^{(\nu)} \;=\; \int_{-\infty}^{\infty} dh \int_{\varepsilon}^{\infty} d\lambda\; \lambda^\nu\; \frac{J_m(\lambda r)J_m(\lambda r')\;e^{ih(z-z')}}{k^2(\lambda^2+h^2)}, \qquad (25.21)$$

The superscript and exponent is an odd integer, $\nu \geq -1$ for $M_m^{(\nu)}$ and $N_m^{(\nu)}$); $\nu \geq 1$ for $L_m^{(\nu)}$). For convenience, the lower limit ε has been chosen in place of zero, so that all the integrals exist. This permits one to draw the operators $\mathbf{T}$ contained in the vector fields under the integral in front of the integral. After appropriate transformations and evaluations of the integrals have been done, these are inserted into eq.(25.18), the operators are again introduced into the integrals. after the derivations contained in the operators have been done, the limit $\varepsilon \to 0$ can be done.

25.3 Representations of the Green's tensor without the irrotational Hansen harmonics

There are several ways to remove the Hansen harmonics from the Green's tensor. We shall show three methods, each one leading to a different representation.

25.3.1 Representation with the unit tensor in the singular term

In the same way as in section 23.8 the irrotational Hansen harmonics may be removed by subtracting the singular term $\mathbf{I}\ \delta(\mathbf{r}-\mathbf{r}')/k^2$ as given in eq.(25.15) from the representation given in (25.16) with (25.17). This gives:

$$\begin{aligned}\mathbf{\Gamma}(\mathbf{r};\mathbf{r}') &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_{m=-\infty}^{\infty} \left(\mathbf{M}_{mh\lambda}(\mathbf{r})\mathbf{M}^*_{mh\lambda}(\mathbf{r}') + \mathbf{N}_{mh\lambda}(\mathbf{r})\mathbf{N}^*_{mh\lambda}(\mathbf{r}')\right) \times \\ &\qquad \times \left(\frac{1}{\lambda(\lambda^2+h^2-k^2)} + \frac{\lambda}{k^2}\,\frac{1}{\lambda^2+h^2}\right) \quad - \\ &\quad - \frac{1}{k^2}\,\mathbf{I}\ \delta(\mathbf{r}-\mathbf{r}'). \end{aligned} \tag{25.22}$$

25.3.2 Representation with the dyadic $\mathbf{e}_z\mathbf{e}_{z'}$

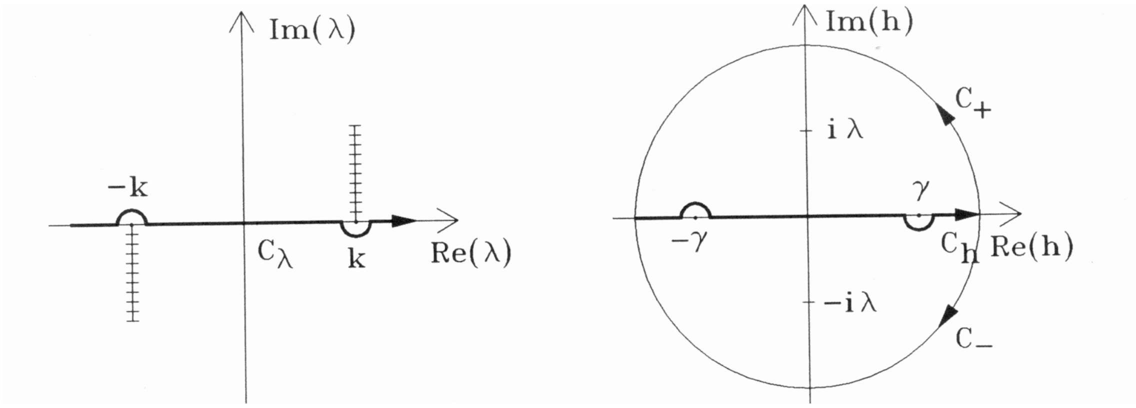

Figure 25.1: **Links**: a) The complex $\lambda-$plane with branch cuts; **Rechts** b) The complex $h-$plane with poles at $\gamma = \pm i\lambda$.

One of the essential steps to obtain new representations of the Green's tensor, in which the irrotational Hansen harmonics no longer occur and which involve less integrations, is to do one integration. If this integration is done w.r.t. the longitudinal wave number h then the resulting integral will have a discontinuous derivative w.r.t. z; and a further derivation w.r.t. z' will yield the distribution $\delta(z-z')$. Performing all the derivations required by the operators bT in (25.18) will lead to some cancellations of terms such that only the solenoidal Hansen harmonics and a singular term porportional to the dyadic $\mathbf{e}_z\mathbf{e}_{z'}$ remain in the resulting representation of the Green's tensor:

$$\begin{aligned}\mathbf{\Gamma}(\mathbf{r};\mathbf{r}') &= \frac{i}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} d\lambda\ \frac{1}{\lambda\gamma}\left[\mathbf{M}^+_{m\gamma\lambda}(\mathbf{r})\mathbf{M}^{+*}_{m\gamma\lambda}(\mathbf{r}') + \mathbf{N}^+_{m\gamma\lambda}(\mathbf{r})\mathbf{N}^{+*}_{m\gamma\lambda}(\mathbf{r}')\right]\Theta(z-z') \\ &+ \frac{i}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} d\lambda\ \frac{1}{\lambda\gamma}\left[\mathbf{M}^-_{m\gamma\lambda}(\mathbf{r})\mathbf{M}^{-*}_{m\gamma\lambda}(\mathbf{r}') + \mathbf{N}^-_{m\gamma\lambda}(\mathbf{r})\mathbf{N}^{-*}_{m\gamma\lambda}(\mathbf{r}')\right]\Theta(z'-z) \\ &- \frac{1}{k^2}\ \delta(\mathbf{r}-\mathbf{r}')\ \mathbf{e}_z\mathbf{e}_{z'}. \end{aligned} \tag{25.23}$$

The Hansen harmonics $\mathbf{M},\mathbf{N}$ have been defined above in eqs.(25.3) and (25.5); for $\mathbf{M}^{\pm},\mathbf{N}^{\pm}$ the function ψ must be replaced by

$$\psi^{\pm}_{m\gamma\lambda} = J_m(\lambda r)\ e^{im\phi}\ e^{\pm i\gamma z}. \tag{25.24}$$

The function γ is defined as:

$$\gamma = \sqrt{k^2-\lambda^2}, \qquad \mathrm{Im}\gamma \geq 0. \tag{25.25}$$

The path of integration C'_λ is shown in Fig.25.1a. $\Theta()$ is the Heaviside unit step function.

In order to find the representation (25.23) the integrals defined in eqs.(25.19) to (25.21) must be evaluated w.r.t. h. The integral is single-valued. The path of integration in the complex h-plane is shown in Fig.25.1b). It is closed by a semi-circle of infinite radius in the upper (lower) ahlf-plane for $(z - z') > 0$ (< 0). Then the integral can be evaluated by the residue theorm. The resulting integrals depend on the modulus $|z - z'|$:

$$
\begin{aligned}
M_m^{(\nu)} &= i\pi \int_\varepsilon^\infty d\lambda \; \frac{\lambda^\nu}{\gamma} \; J_m(\lambda r) J_m(\lambda r') \; e^{i\gamma|z-z'|}, & (25.26)\\
N_m^{(\nu)} &= \frac{i\pi}{k^2} \int_\varepsilon^\infty d\lambda \; \lambda^\nu \; J_m(\lambda r) J_m(\lambda r') \left[\frac{e^{i\gamma|z-z'|}}{\gamma} \; - \; \frac{e^{-\lambda|z-z'|}}{i\lambda} \right], & (25.27)\\
L_m^{(\nu)} &= \frac{\pi}{k^2} \int_\varepsilon^\infty d\lambda \; \lambda^\nu \; J_m(\lambda r) J_m(\lambda r') \; e^{-\lambda|z-z'|}. & (25.28)
\end{aligned}
$$

$L_m^{(\nu)}$, $\nu \geq 1$ cotains only modes evanescent in the z-direction, which result from the poles at $h = \pm\, i\lambda$. Their contribution are cancelled by corresponding terms arising from $N_m^{(\nu)}$ after the integrals (**??**) to (**??**) have been inserted into the representation (**??**) and the operators **T** have been shifted into the integrals. Derivatives w.r.t. z and/or z' of the exponential function depending on $|z - z'|$ are:

$$
\begin{aligned}
\partial_{z'} \; e^{\beta|z-z'|} &= - \, \partial_z \; e^{\beta|z-z'|} \; = \; \beta \; e^{\beta|z-z'|} \; \mathrm{sign}(z' - z), & (25.29)\\
\partial_{z'}\partial_z \; e^{\beta|z-z'|} &= - \, \beta^2 \; \beta|z - z'| \; - \; 2\,\beta \; \delta(z - z'). & (25.30)
\end{aligned}
$$

It will suffice to calculate two elements, $\mathbf{G}_{zz'}$ and $\mathbf{G}_{rr'}$, to show how this approach works.

$$
\begin{aligned}
\mathbf{G}_{mzz'} &= \left[0 \; + \; N_m^{(3)} \; - \; \frac{2\pi}{k^2} \; \delta(z - z') \int_0^\infty \lambda d\lambda \; J_m(\lambda r) J_m(\lambda r') \right] \; e^{im(\phi-\phi')} & (25.31)\\
&= \left[0 \; + \; N_m^{(3)} \; - \; \frac{2\pi}{k^2} \; \delta(z - z') \; \frac{\delta(r - r')}{r} \right] \; e^{im(\phi-\phi')}. & (25.32)
\end{aligned}
$$

The zero recalls that the z-component of Hansen harmonics of type **M** are zero. The last factor yields the distribution $\delta(\phi - \phi')$ as soon as the summation over m is done; this completes the singular term.

$$
\mathbf{G}_{zz'} \; = \; \frac{i}{4\pi \; k^2} \sum_{m=-\infty}^{\infty} \int_0^\infty d\lambda \; \lambda^3 \; J_m(\lambda r) J_m(\lambda r') \; e^{i\gamma|z-z'|} \; e^{im(\phi-\phi')} \; - \; \frac{1}{k^2} \; \delta(\mathbf{r} - \mathbf{r}'). \qquad (25.33)
$$

Thus this element can be rewritten as given in representation (25.23).

In the element $\mathbf{G}_{mrr'}$ all terms with evanescent mode contributions cancel as well as terms containing $\delta(z - z')$.

$$
\begin{aligned}
\mathbf{G}_{mrr'} \; = \; & i\pi \int_0^\infty d\lambda \; \frac{1}{\lambda\gamma} \; \left(\frac{1}{r} \partial_\phi \; J_m(\lambda r) \; e^{im\phi} \right) \left(\frac{1}{r'} \partial_{\phi'} \; J_m(\lambda r') \; e^{-im\phi'} \right) \; e^{i\gamma|z-z'|} \\
+ \; & \frac{i\pi}{k^2} \int_0^\infty d\lambda \; \frac{\gamma}{\lambda} \; \left(\frac{1}{r} \partial_r \; J_m(\lambda r) \; e^{im\phi} \right) \left(\frac{1}{r'} \partial_{r'} \; J_m(\lambda r') \; e^{-im\phi'} \right) \; e^{i\gamma|z-z'|}. \qquad (25.34)
\end{aligned}
$$

Using the definitions (25.3) and (25.5) of the operators **T** the last expression is transformed to that given in eq.(25.23). The transformation and evaluation of the other elements of (25.18) required to obtain the representation (25.23) goes along the same lines.

25.3.3 Representation with the dyadic $\mathbf{e}_r\mathbf{e}_{r'}$

In a way similar to that in the preceeding subsection a new representation of the Green's tensor can be obtained which consists of solenoidal Hansen harmonics and a singular term proportional to the dyadic $\mathbf{e}_r\mathbf{e}_{r'}$. But befor the integration w.r.t. the radial wave number λ can be done the integral must be transformed ionto one over the whole real λ-axis; this implies that the Bessel functions of the first kind, $J_m(\lambda r_>)$, is replaced with the Hankel function of the first kind, $H^1_m(\lambda r_>)$.

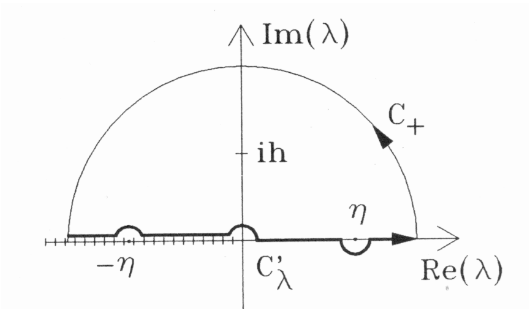

Figure 25.2: The complex $\lambda-$plane with branch cut.

The representation of the Green's tensor with a radially directed singular term is:

$$\begin{aligned}\mathbf{G}(\mathbf{r};\mathbf{r}') &= \frac{i}{8\pi}\sum_{m=-\infty}^{\infty}\int_{C_h} dh\,\frac{\mathbf{M}_{mh\eta}(\mathbf{r}_<)\mathbf{M}^{2*}_{mh\eta}(\mathbf{r}_>)\;+\;\mathbf{N}_{mh\eta}(\mathbf{r}_<)\mathbf{N}^{2*}_{mh\eta}(\mathbf{r}_>)}{\eta^2}\\ &- \frac{1}{k^2}\,\delta(\mathbf{r}-\mathbf{r}')\;\mathbf{e}_r\mathbf{e}_{r'}\end{aligned} \tag{25.35}$$

with

$$\eta \;=\; \sqrt{k^2-h^2}, \qquad \mathrm{Im}(\eta)\geq 0. \tag{25.36}$$

$\mathbf{r}_>, (\mathbf{r}_<)$ is that of the triples (r,θ,ϕ) and (r',θ',ϕ') in which the first variable is max(r,r') (min(r, r')). The vector fields $\mathbf{M},\mathbf{N},\mathbf{L}$ are defined in eqs.(25.3), (25.5), (25.7). The corresponding fields with a superscript (1) or (2) are obtaind from these definitions by replacing in (25.1) the Bessel function of the first kind by the Hankel function of the first or second kind. Note that complex conjugation transforms the Hankel function of the first kind into that of the second kind and vice versa. The path of integration, C_h in the complex h-plane and the branch cuts are the same as in Fig.1(a) with λ replaced by h.

In the derivation of the representation (25.35) the integrals $M^{(nu)}_m, N^{(nu)}_m, L^{(nu)}_m$ defined in eqs.() to () must be transformed such that the integration over the radial wave number λ extends over the whole real $\lambda-$axis (cf. Fig.24.2); then this integration can be done by the residue theorem. The exponent of λ is assumed to be an odd integer $\nu\geq -1$. We start from an integral I extending over the whole $\lambda-$axis, in which $f(\lambda)$ is to be identified with one of the following three functions even in λ.

$$f(\lambda) \;=\; \left(\frac{1}{(\lambda^2+h^2-k^2)(\lambda^2+h^2)};\;\frac{1}{(\lambda^2+h^2-k^2)};\;\frac{1}{(\lambda^2+h^2)}\right)$$

$$\begin{aligned} I \;&=\; \tfrac{1}{2}\int_{-\infty}^{-\varepsilon} d\lambda\;\lambda^\nu\;f(\lambda)\;J_m(\lambda r_<)H^{(1)}_m(\lambda r_>) \;+\; \tfrac{1}{2}\int_{\varepsilon}^{\infty} d\lambda\;\lambda^\nu\;f(\lambda)\;J_m(\lambda r_<)H^{(1)}_m(\lambda r_>) \;=\\ &=\; \tfrac{1}{2}\int_{\infty}^{\varepsilon} d\lambda\;\lambda^\nu\;e^{i\pi\nu}\;f(\lambda)\;J_m(\lambda e^{i\pi}r_<)H^{(1)}_m(\lambda e^{i\pi}r_>) \;+\; \ldots \;=\\ &=\; \tfrac{1}{2}\int_{\varepsilon}^{\infty} d\lambda\;\lambda^\nu\;f(\lambda)\;J_m(\lambda r_<)H^{(2)}_m(\lambda r_>) \;+\; \ldots \;=\\ &=\; \int_{\varepsilon}^{\infty} d\lambda\;\lambda^\nu\;f(\lambda)\;J_m(\lambda r_<)\;J_m(\lambda r_>)\end{aligned} \tag{25.37}$$

In the first integral of the second line $e^{i\pi\nu} = -1$ since ν is odd; this sign is used in the next line to reverse the integration limits. In this integral the half circuit relation of the Bessel functions

$$J_m(\lambda e^{i\pi} r_<) H_m^{(1)}(\lambda e^{i\pi} r_>) \;=\; (-1)^m \; J_m(\lambda r_<) H_m^{(2)}(\lambda r_>) \; (-1)^m$$

is used to simplify the arguments. The connection between the Bessel function of the first kind and the Hankel functions

$$J_m(\lambda r_<) \;=\; \frac{1}{2} \left[H_m^{(1)}(\lambda r) + H_m^{(2)}(\lambda r) \right]$$

is applied in going to the last line. The denominators of the functions $f(\lambda)$ require the path of integration C'_λ to be indented at $\lambda = \pm \sqrt{k^2 - h^2} = \pm \eta$ as indicated in Fig.24.2. In the limit $\varepsilon \to 0$ the intgral in athe last line of (25.37) ceases to exist if both m = 0 and $\nu = -1$. A careful check of the various terms in eq.(25.18) reveals that there is no term where this happens; the operators **T**produce either additional factors of λ or a factor m = 0. In the limit $\varepsilon \to 0$ the integral in the first line of (25.37) may be regarded as a Cauchy principal value integral. In view of the behavior of Bessel functions for small argument