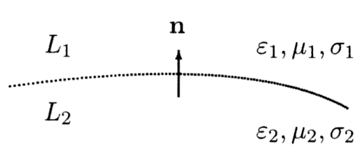
Chapter 19

Green's Function for Problems with Layers

19.1 Continuity conditions

The medium comprises two or more straight or curved layers. The properties of the medium are constant within each layer; they change discontinuously across the interface separating two layers. Within each layer the phenomena are described by a differential equation; the latter cease to be valid in the interface. There they are replaced by continuity conditions. For example in electrostatics, magnetostatics and electrodynamics the field components must fulfil the following continuity conditions: The tangential components of the electric and of the magnetic field must be continuous as well as the normal components of the dielectric displacement and of the magnetic induction. These conditions are cast into the following formulas:



$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0, \qquad \vec{E}_{t1} = \vec{E}_{t2}; \tag{19.1}$$

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0, \qquad \vec{H}_{t1} = \vec{H}_{t2}; \tag{19.2}$$

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0, \qquad \vec{D}_{t1} = \vec{D}_{t2}; \tag{19.3}$$

$$\vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0, \qquad \vec{B}_{t1} = \vec{B}_{t2}. \tag{19.4}$$

19.2 Green's functions for two-layer problems

It is convenient though not obligatory to ornate the scalar Green's function with two subscripts:

$$G_{ik}(\vec{r}, \vec{r}\,') \qquad \begin{array}{l} i \text{ label of layer containing the field point} \\ k \text{ label of layer containing the source point} \end{array}$$

This is written out in more detail for a problerm with two layers as:

$$\begin{aligned} G_{11}(\vec{r}, \vec{r}\,'): &\qquad \vec{r} \in L_1, \qquad \vec{r}\,' \in L_1; \\ G_{12}(\vec{r}, \vec{r}\,'): &\qquad \vec{r} \in L_1, \qquad \vec{r}\,' \in L_2; \\ G_{21}(\vec{r}, \vec{r}\,'): &\qquad \vec{r} \in L_2, \qquad \vec{r}\,' \in L_1; \\ G_{22}(\vec{r}, \vec{r}\,'): &\qquad \vec{r} \in L_2, \qquad \vec{r}\,' \in L_2. \end{aligned} \tag{19.5}$$

It is essential to write the differential equations with selfadjoint operators. Even if the parameters describing the physical properties of the layers are constant within the layers, one should write the differential equations as if these parameters were functions of the space coordinates. If there are n layers then the labels i, k run as 1, 2, ..., n .

19.2.1 The electrostatic two-layer problem

The Poisson equation for the potential in a problem with a layered dielectric should be written as:

$$\operatorname{div}\big(\varepsilon(\vec r)\operatorname{grad}\Phi\big) \;=\; -\,\rho(\vec r). \tag{19.6}$$

The corresponding differential equation for the Green's function for two layers each consisting of a constant dielectric (labelled as $\varepsilon_1,\ \varepsilon_2$ respectively) is:

$$\nabla\big(\varepsilon_i\cdot\nabla G_{ik}\big) \;=\; -\,\delta_{ik}\;\delta(\vec r-\vec r\,'). \tag{19.7}$$

There is no summation over the repeated subscript i ! The pieces of the Green's function having equal subscripts are the solutions of inhomogeneous differential equations, those bearing different subscripts are solutions of homogeneous equations. Still, the latter, too, are determined uniquely by the continuity conditions. In the present case these are eqs.(19.1) and (19.3). We assume that the potential consists also of two pieces, each one valid in the corresponding layer. The boundary conditions then connect these two pieces and their normal derivatives. The relations are then taken over for the pieces of the Green's function. The points of the interface are denoted as $\vec r_{12}$. So we get from the first condition:

$$\vec E_i \;=\; -\,\nabla\Phi_i, \qquad \vec E_{it} \;=\; -\,\nabla_t\Phi_i. \tag{19.8}$$

For points in the interface $\vec r_{12}$ we find:

$$\begin{aligned}\vec r\;\in\;\vec r_{12}: \qquad & \vec E_{it} \;=\; -\,\nabla_t\Phi_i \;=\; \vec E_{it} \;=\; -\,\nabla_t\Phi_i,\\ \Rightarrow\quad & \Phi_1 \;=\; \Phi_2 \quad\Rightarrow\quad \underline{G_{1k} \;=\; G_{2k}, \quad k=1,2}.\end{aligned} \tag{19.9}$$

Condition (19.3) gives:

$$\begin{aligned}\vec r\;\in\;\vec r_{12}: \qquad & D_{1n} \;=\; -\,\varepsilon_1\,\frac{\partial\Phi_{1n}}{\partial n} \;=\; D_{2n} \;=\; -\,\varepsilon_2\,\frac{\partial\Phi_{2n}}{\partial n},\\ \Rightarrow\quad & \varepsilon_1\,\frac{\partial\Phi_1}{\partial n} \;=\; \varepsilon_2\,\frac{\partial\Phi_2}{\partial n} \quad\Rightarrow\quad \underline{\varepsilon_1\,\frac{\partial G_{1k}}{\partial n} \;=\; \varepsilon_2\,\frac{\partial G_{2k}}{\partial n}, \quad k=1,2.}\end{aligned} \tag{19.10}$$

19.2.2 The source representation

In order to derive the source representation, it is most convenient to start where the dependence of the dielectric constant and of the Green's function is taken into account in the dependence on the variables. At the end, the constancy of the dielectric constant in each layer is shown by constants with subscripts. Then also the other quantities are provided with the corresponding labels. We start from Green's second theorem in the generalized form for the primed variable and we identify one function with the potential $\Phi(\vec r\,')$ the other one with $G(\vec r,\vec r\,')$:

$$\begin{aligned}&\int\!\!\int\!\!\int G(\vec r\,',\vec r)\;\underbrace{\nabla'\big(\varepsilon(\vec r\,')\cdot\nabla'\Phi(\vec r\,')\big)}_{=-\;\rho(\vec r\;r')}\;d\vec r\,' \;-\int\!\!\int\!\!\int \Phi(\vec r\,')\underbrace{\nabla'\big(\varepsilon(\vec r\,')\cdot\nabla G(\vec r\,',\vec r)\big)}_{=-\;\delta(\vec r-\vec r\,')}\;d\vec r\,' \;=\\ &=\;\int\!\!\int\; dF'\vec n\cdot\big(G(\vec r\,',\vec r)\;\varepsilon(\vec r\,')\nabla'\Phi(\vec r\,')\big) \;-\; \int\!\!\int\; dF'\vec n\cdot\big(\Phi(\vec r\,')\varepsilon(\vec r\,')\nabla' G(\vec r\,',\vec r)\big).\end{aligned}$$

We do the integration over the delta-distribution and use the symmetry of the scalar Green's function. Rearranging the resulting equation gives:

$$\begin{aligned}\Phi(\vec r) \;&=\; \int\!\!\int\!\!\int G(\vec r,\vec r\,')\;\rho(\vec r\,')\;d\vec r\,' \;+\\ &+\;\int\!\!\int dF'\;\varepsilon(\vec r\,')\left(G(\vec r,\vec r\,')\;\frac{\partial\Phi(\vec r\,')}{\partial n'} \;-\; \Phi(\vec r\,')\frac{\partial G(\vec r,\vec r\,')}{\partial n'}\right).\end{aligned} \tag{19.11}$$

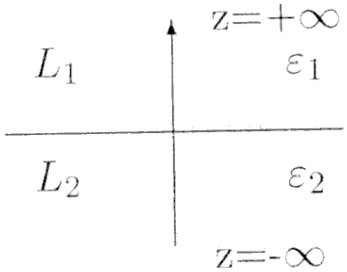
Now we assume that the charge density ρ is given in a volume V extending over both layers. The surface F_{12} is in the interface separating the two layers; it splits the volume V into a part V_1 lying in layer 1 and a part V_2 lying in layer 2. The surfaces S_1, S_2 respectively cover the volumes V_1, V_2 respectively so that $S_1 + S_{12}$ encloses V_1; similarly $S_2 + S_{12}$ encloses V_2. Now we specialize the dielectric constants as being constant in each layer. We use the labels belonging to the layers as defined for the Green's function in eqs.(19.5) and we attach corresponding labels to Φ and ρ. So we get:

$$\begin{aligned}
\Phi_1(\vec{r}) \;&=\; \int\int_{V_1}\int G_{11}(\vec{r},\vec{r}\,')\;\rho_1(\vec{r}\,')\;d\vec{r}\,' \;+\; \int\int_{V_2}\int G_{12}(\vec{r},\vec{r}\,')\;\rho_2(\vec{r}\,')\;d\vec{r}\,' \;+ \\
&+\int_{F_1}\int \;dF'\;\varepsilon_1\left(G_{11}(\vec{r},\vec{r}\,')\;\frac{\partial\Phi_1(\vec{r}\,')}{\partial n'} \;-\; \Phi_1(\vec{r}\,')\frac{\partial G_{11}(\vec{r},\vec{r}\,')}{\partial n'}\right) \;+ \\
&+\int_{F_2}\int \;dF'\;\varepsilon_2\left(G_{12}(\vec{r},\vec{r}\,')\;\frac{\partial\Phi_2(\vec{r}\,')}{\partial n'} \;-\; \Phi_2(\vec{r}\,')\frac{\partial G_{12}(\vec{r},\vec{r}\,')}{\partial n'}\right).
\end{aligned} \tag{19.12}$$

$$\begin{aligned}
\Phi_2(\vec{r}) \;&=\; \int\int_{V_1}\int G_{21}(\vec{r},\vec{r}\,')\;\rho_1(\vec{r}\,')\;d\vec{r}\,' \;+\; \int\int_{V_2}\int G_{22}(\vec{r},\vec{r}\,')\;\rho_2(\vec{r}\,')\;d\vec{r}\,' \;+ \\
&+\int_{F_1}\int \;dF'\;\varepsilon_1\left(G_{21}(\vec{r},\vec{r}\,')\;\frac{\partial\Phi_1(\vec{r}\,')}{\partial n'} \;-\; \Phi_1(\vec{r}\,')\frac{\partial G_{21}(\vec{r},\vec{r}\,')}{\partial n'}\right) \;+ \\
&+\int_{F_2}\int \;dF'\;\varepsilon_2\left(G_{22}(\vec{r},\vec{r}\,')\;\frac{\partial\Phi_2(\vec{r}\,')}{\partial n'} \;-\; \Phi_2(\vec{r}\,')\frac{\partial G_{22}(\vec{r},\vec{r}\,')}{\partial n'}\right).
\end{aligned} \tag{19.13}$$

19.2.3 Two dielectric half-spaces separated by a plane interface

The formulas of the previous section are specialized to this case (no summation over i)



$$\varepsilon_i \Delta G_{ik} \;=\; -\;\delta_{ik}\;\delta(\vec{r}-\vec{r}\,'). \tag{19.14}$$

$$G_{1k} \;=\; G_{2k}, \tag{19.15}$$

$$\varepsilon_1\frac{\partial G_{1k}}{\partial z} \;=\; \varepsilon_2\frac{\partial G_{2k}}{\partial z}; \tag{19.16}$$

$$\lim_{z\to+\infty} G_{1k} = 0, \qquad \lim_{z\to-\infty} G_{2k} = 0. \tag{19.17}$$

There are several methods to calculate this Green's function. One uses image charges. Another one uses integral representations including the Sommerfeld integral.

Computing the Green's function by image charges

The problem is a standard exercise in electrostatics as an example that the method of image charges well-known for ideally conducting planes of infinite extent works also for dielectric half spaces, e.g. [19.1]. We introduce cylindrical coordinates We assume a source point location $\rho' = 0, z = z'$. The four pieces of the corresponding Green's function are:

$$G_{11}(\rho, z, \rho' = 0, z') \;=\; \frac{1}{4\pi\varepsilon_1}\left[\frac{1}{R_1} - \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\frac{1}{R_2}\right], \tag{19.18}$$

$$G_{22}(\rho, z, \rho' = 0, z') \;=\; \frac{1}{4\pi\varepsilon_2}\left[\frac{1}{R_2} - \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\frac{1}{R_1}\right], \tag{19.19}$$

$$G_{ij}(\rho, z, \rho' = 0, z') \;=\; \frac{1}{2\pi\varepsilon_2}\frac{1}{\varepsilon_1+\varepsilon_2}\frac{1}{R_j}, \quad (i \neq j); \tag{19.20}$$

$$R_{1,2} \;=\; \sqrt{\rho^2 \;+\; (z \mp z')^2}. \tag{19.21}$$

For a source position ρ', ϕ' off the z-axis, one need only replace ρ^2 by $\rho^2 - 2\rho\rho' \cos(\phi - \phi') + \rho^{2'}$ in the last expressions above. Details of the calculations are presented in the notebook quoted in §19.4.1.

Computing the Green's function by integral representations

The Sommerfeld integral (15.30) is used to represent the source term:

$$\frac{1}{\sqrt{\rho^2 + c^2}} = \int_0^\infty e^{-\zeta|c|}\ J_0(\zeta\rho)\ d\zeta. \tag{19.22}$$

Correspondingly, the following integral representations are set up for the four pieces of the Green's function:

$$G_{ii} = \frac{1}{4\pi\varepsilon_i}\int_0^\infty J_0(\zeta\rho)\ \left[e^{-\zeta|z\mp z'|}\ +\ h_{ii}(\zeta)\ e^{\mp\zeta z}\right]\ d\zeta, \tag{19.23}$$

$$= \frac{1}{4\pi\varepsilon_i}\int_0^\infty J_0(\zeta\rho)\ g_{ii}(\zeta)\ d\zeta, \tag{19.24}$$

$$G_{ij} = \frac{1}{4\pi}\int_0^\infty J_0(\zeta\rho)\ g_{ij}(\zeta)\ e^{\mp\zeta z}\ d\zeta; \quad g_{ij}(\zeta) = h_{ij}(\zeta). \tag{19.25}$$

The signs of the exponentials have been chosen to satisfy the boundary conditions at $z = \pm\infty$, eqs.(19.17), the minus sign for i =1, the plus for i = 2. These expressions are inserted into the continuity conditions (19.15) and (19.16). The resulting linear equations for the four amplitudes h_{ij} are solved and the amplitudes are inserted into the integrals given above. So we get:

$$G_{11}(\rho, z, \rho' = 0, z') = \frac{1}{4\pi\varepsilon_1}\int_0^\infty d\zeta\ J_0(\zeta\rho)\ \left[e^{-\zeta|z-z'|}\ +\ \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}e^{-\zeta(z+z')}\right], \tag{19.26}$$

$$G_{12}(\rho, z, \rho' = 0, z') = \frac{1}{4\pi}\frac{2}{\varepsilon_1 + \varepsilon_2}\int_0^\infty d\zeta\ J_0(\zeta\rho)\ e^{-\zeta(z+|z'|)}, \tag{19.27}$$

$$\tag{19.28}$$

$$G_{21}(\rho, z, \rho' = 0, z') = \frac{1}{4\pi}\frac{2}{\varepsilon_1 + \varepsilon_2}\int_0^\infty d\zeta\ J_0(\zeta\rho)\ e^{-\zeta(|z|+z')}, \tag{19.29}$$

$$\tag{19.30}$$

$$G_{22}(\rho, z, \rho' = 0, z') = \frac{1}{4\pi\varepsilon_2}\int_0^\infty d\zeta\ J_0(\zeta\rho)\ \left[e^{-\zeta|z+z'|}\ -\ \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}e^{-\zeta(z+z')}\right]. \tag{19.31}$$

The calculation and the results are presented in the notebook displayed in AnMe14-4-2.pdf. All the above integrals may be transformed into closed expressions with the help of the Sommerfeld integral (19.22). The resulting expression agree with those given in eqs.(19.18) - (19.21).

19.3 Green's function for the three-dimensional potential equation with two layers

We consider the three-dimensional potential equation for a problem with two plane layers. The thickness of the upper (lower) layer is p (q), s. Fig.19.3. The Green's function is a scalar comprising four pieces distinguished by the two labels i, k. The first (second) label denotes the layer containing the point of observation (the source point).

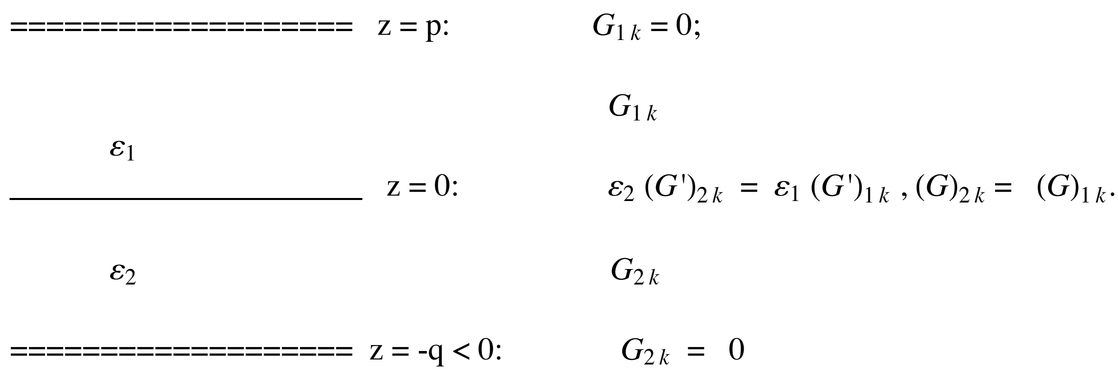


Figure 19.1: An infinite plane condensor comprising two layers with permeability ε_1 and ε_2. It is bounded by planes of ideal conductivity. The primes of the continuity conditions at $z = 0$ denote derivations with respect to z.

The Green's function $G_{ik}(\vec{r}, \vec{r}\,')$ is a solution of the following inhomogeneous differential equation :

$$\varepsilon_i \Delta G_{ik}(\vec{r}, \vec{r}\,') = -\,\delta_{ik}\;\delta(\vec{r} - \vec{r}\,') \tag{19.32}$$

$$p \geq z, z' \geq -q, \qquad -\infty < x, x' < \infty, \quad -\infty < y, y' < \infty. \tag{19.33}$$

The Green's function is represented as a Fourier integral in the transverse directions x, y:

$$G_{ik}(\vec{r}, \vec{r}\,') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_x e^{i(k_x(x-x') + k_y(y-y'))}\; g_{ik}(k_x, k_y; z, z'). \tag{19.34}$$

Inserting this and the Fourier integral representations of the Delta-distribution in the transverse coordinates gives the following differential equation for the amplitude function $g_{ik}(k_x, k_y; z, z') = g_{ik}(\kappa; z, z')$:

$$\varepsilon_i \left(\frac{d^2}{dz^2} \;-\; \kappa^2 \right) \bar{g}_{ik}(\kappa; z, z') = -\;\delta_{ik}\;\delta(z - z') \tag{19.35}$$

with

$$\kappa = \sqrt{k_x^2 + k_y^2}. \tag{19.36}$$

A particular solution of the inhomogeneous equations is:

$$\bar{g}_{ik}(\kappa; z, z') = \delta_{ik}\; \frac{1}{2\kappa\varepsilon_i}\; e^{-\kappa|z-z'|}. \tag{19.37}$$

The general solution for each piece of the Green's function is:

$$\bar{g}_{ik}(\kappa; z, z') = \delta_{ik}\; \frac{1}{2\kappa\varepsilon_i}\; e^{-\kappa|z-z'|} \;+\; \frac{a_{ik}}{\kappa}\; e^{\kappa z} \;+\; \frac{b_{ik}}{\kappa}\; e^{-\kappa z}. \tag{19.38}$$

These solutions are inserted into the integral representation (19.34). In addition, plane polar coordinates are introduced for the transverse variables in coordinate and in k-space:

$$x = \rho\;\cos\phi, \qquad y = \rho\;\sin\phi; \tag{19.39}$$

$$x' = \rho'\;\cos\phi', \quad y' = \rho'\;\sin\phi'; \tag{19.40}$$

$$k_x = \kappa\;\cos\psi, \quad k_y = \kappa\;\cos\psi. \tag{19.41}$$

$$R = \sqrt{(x - x')^2 \;+\; (y - y')^2} = \sqrt{\rho^2 - 2\rho\rho'\cos(\phi - \phi') + \rho'^2}. \tag{19.42}$$

In place of the integral representation (19.34) we get:

$$G_{ik}(\vec{r},\vec{r}\,') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\kappa\ \kappa \int_0^{2\pi} d\psi\ e^{i\kappa R\cos(\phi-\phi'-\psi)}\bar{g}_{ik}(\kappa;z,z'), \tag{19.43}$$

$$G_{ik}(\rho,\phi,z;\rho',\phi',z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\kappa\ J_0(\kappa R)\ g_{ik}(\kappa;z,z'), \tag{19.44}$$

$$g_{ik}(\kappa;z,z') = \delta_{ik}\ \frac{1}{2\varepsilon_i}\ e^{-\kappa|z-z'|}\ +\ a_{ik}\ e^{\kappa z}\ +\ b_{ik}\ e^{-\kappa z}. \tag{19.45}$$

The integral over ψ is an integral representation of the Bessel function $J_0(\kappa R)$:

$$J_0(\kappa R)\ =\ \frac{1}{2\pi}\int_0^{2\pi} d\psi\ e^{i\kappa R\cos(\phi-\phi'-\psi)}. \tag{19.46}$$

If the source is on the $z'-$axis then $R=\rho$. Most of the derivations below will be done for this special case. The transition to the general case just requires the replacement $r\rightarrow R$.

19.3.1 An integral representation for the four pieces of the Green's function

In preparation.

19.3.2 Expressions for the cured Green's function

In preparation.

19.4 Further examples of Green's functions for problems with several layers

The approach to find Green's functions in problems with several layers explained in the first sections of this chapter has been used to treat models of counter configurations comprising up to three layers [19.2]. The corresponding Green's functions encounter convergence troubles if the field point is in the same plane as the source point. Similar difficulties arise also from image points. All these problems are cured by subtracting terms in the integral representations to compensate convergence problems.

19.5 *Mathematica* notebooks

19.5.1 Point charge q with two dielectrica: Method of images

AnMe19-4-1.pdf , 6 pages.

19.5.2 Greens function for two dielectric halfspaces. Method of integral representations

Greens function for two dielectric halfspaces. Computation of the amplitudes in the integral representations of the four pieces.
AnMe19-4-2.pdf , 4 pages.

19.5.3 Greens function for a condensor with two layers. Convergence acceleration by removing slowly convergent terms in the integral representation

In preparation .

19.6 References

[19.1] J.D. Jackson, Classical Electrodynamics. 3rd. ed., Wiley, 1998.
[19.2] Th. Heubrandtner, B. Schnizer, C. Lippmann, W. Riegler, Static electric fields in an infinite plane condensor with one or three homogeneous layers. Nucl. Instrum. Methods Phys. Res., A : 489 (2002) no.1-3, pp.439-443 (= CERN-EP-2002-004 ; Geneva : CERN , 8 Jan 2002)
[19.3] Th. Heubrandtner, B. Schnizer, C. Lippmann, W. Riegler, Static electric fields in an infinite plane condensor with one or three homogeneous layers. Report CERN-OPEN-2001-074, 31 Oct. 2001 (This is an extended preprint version of the previous paper)