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ON THE SUMMATION OF INFINITE ALGEBRAIC AND FOURIER SERIES

by

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Abstract

General analytic expressions have been derived for the summation of doubly infinite sums of terms consisting of the ratio of two polynomials, or of the product of such a ratio with trigonometric functions. Although many special cases of such sums are known, the formalism can be applied to cases that are difficult, if at all, available in the literature.

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INTRODUCTION

The summation of infinite series can become quite time consuming and expensive even with high-speed computers if their convergence is slow. Although a number of books exist which have collections of analytically summable series [1-4], there seems to be a lack of descriptions of general methods which permit the summation of series which are not listed.

In this contribution we present a few general formulae which appear to be quite useful in practice, but could not be found in any of the more common books on series. Since their derivation is quite straightforward from decomposition into partial fractions, it is quite possible that these formulae have been derived before. However, their applications are so numerous that it appears useful to make them known to a wider audience at the peril of repetition.

1. Infinite series consisting of terms which are ratios of two polynomials, and where the summation index runs from $-\infty$ to $+\infty$, can be summed with the help of the expressions

$$\sum_{n=-\infty}^{+\infty} \frac{Q(n)}{P(n)} = -\pi \sum_{n=1}^{K} \frac{Q(p_n)}{P'(p_n)} \cot \pi p_n \tag{1}$$

$$\sum_{n=-\infty}^{+\infty} (-)^n \frac{Q(n)}{P(n)} = -\pi \sum_{n=1}^{K} \frac{Q(p_n)}{P'(p_n)} \operatorname{cosec} \pi p_n \tag{2}$$

where p_n are the distinct zeros of the polynominal P of order K, and the order of polynomial Q is less than K for convergence.

2. If the polynomial P has multiple roots, the equations (1) and (2) can be generalized as shown in the Appendix. However, it is often more convenient to obtain the result by a limiting process from the expressions for distinct zeros.

If Q/P is an even function of n, it is also possible to obtain the sums from 1 to ∞ by splitting off the terms with negative summation index and inverting the sign. Some simple examples of the applications of the above equations are shown in Table I (most or all of these can be found in standard text-books).

3. A more general expression has been derived which sums trigonometric series with coefficients consisting of ratios of polynomials, and with the summation index running from $-\infty$ to $+\infty$

$$\sum_{n=-\infty}^{+\infty} \frac{Q(n)}{P(n)} \cos n\theta = -\pi \sum_{n=1}^{K} \frac{Q(p_n)}{P'(p_n)} \frac{\cos p_n(\pi - \theta)}{\sin \pi p_n} \qquad (3)$$

$$\sum_{n=-\infty}^{+\infty} \frac{Q(n)}{P(n)} \sin n\theta = +\pi \sum_{n=1}^{K} \frac{Q(p_n)}{P'(p_n)} \frac{\sin p_n(\pi - \theta)}{\sin \pi p_n} \qquad (4)$$

where again p_n are the (distinct) zeros of the polynomial P of order K, and the polynomial Q has an order less than K.

4. The validity of the sine series is restricted to $0 < \theta < 2\pi$ but the cosine series is also valid for $\theta = 0$ (or 2π), where it yields Eq. (1). (For $\theta = \pi$, it yields Eq. (2).) The two trigonometric series can be combined into a single exponential series

$$\sum_{n=-\infty}^{+\infty} \frac{Q(n)}{P(n)} e^{in\theta} = -\pi \sum_{n=1}^{K} \frac{Q(p_n)}{P'(p_n)} \frac{e^{ip_n(\theta-\pi)}}{\sin \pi p_n} \qquad (5)$$

valid for $0 < \theta < 2\pi$. Some simple examples of applications of these formulae are given in Table II.

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REFERENCES

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4) E.R. Hansen; A Table of Series and Products, Prentice Hall (1975).

Table I

$$\sum_{-\infty}^{+\infty} \frac{1}{(n-a)(n-b)} = \frac{\pi}{b-a} (\cot \pi a - \cot \pi b)$$

$$\sum_{-\infty}^{+\infty} \frac{1}{n^2 - a^2} = -\frac{\pi}{a} \cot \pi a$$

$$\sum_{-\infty}^{+\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$$

$$\sum_{-\infty}^{+\infty} \frac{1}{(n\pm a)^2 + b^2} = \frac{\pi}{2b} \frac{\sinh 2\pi b}{\cosh^2 \pi b - \cos^2 \pi a}$$

$$\sum_{-\infty}^{+\infty} \frac{1}{(n\pm a)^2 - b^2} = \frac{\pi}{2b} \frac{\sin 2\pi b}{\cos^2 \pi b - \cos^2 \pi a}$$

$$\sum_{-\infty}^{+\infty} \frac{1}{(n-a)(n-b)(n-c)} = -\pi \left[\frac{\cot \pi a}{(a-b)(a-c)} + \frac{\cot \pi b}{(b-a)(b-c)} + \frac{\cot \pi c}{(c-a)(c-b)}\right]$$

$$\sum_{-\infty}^{+\infty} \frac{n^{(2)}}{(n-a)(n-b)(n-c)} = -\pi \left[\frac{a^{(2)} \cot \pi a}{(a-b)(a-c)} + \frac{b^{(2)} \cot \pi b}{(b-a)(b-c)} + \frac{c^{(2)} \cot \pi c}{(c-a)(c-b)}\right]$$

$$\sum_{-\infty}^{+\infty} \frac{1}{(n\pm a)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

$$\sum_{-\infty}^{+\infty} \frac{1}{(n\pm a)^3} = \pm \frac{\pi^3}{2} \frac{\cos \pi a}{\sin^3 \pi a}$$

$$\sum_{-\infty}^{+\infty} \frac{(-)^n}{n^2 - a^2} = -\frac{\pi}{a \sin \pi a}$$

$$\sum_{-\infty}^{+\infty} \frac{(-)^n}{(n\pm a)^2 - b^2} = \frac{\pi}{b} \frac{\cos \pi a \cdot \sin \pi b}{\cos^2 \pi b - \cos^2 \pi a}$$

$$\sum_{-\infty}^{+\infty} \frac{(-)^n}{(n\pm a)^2} = \pi^2 \frac{\cos \pi a}{\sin^2 \pi a}$$

$$\sum_{-\infty}^{+\infty} \frac{n}{(n-a)(n-b)} = \frac{\pi}{b-a} (a \cot \pi a - b \cot \pi b)$$

$$\sum_{-\infty}^{+\infty} \frac{n}{n^2 \pm a^2} = 0$$

(3) For multiple zeros of the polynomial P(x), the partial fraction representation of Q(x)/P(x) will contain higher powers of one or more terms

$$\frac{Q(x)}{P(x)} = \sum_{n=1}^{K} \frac{a_n}{x - p_n} + \sum_{n=1}^{\ell} \frac{b_n}{(x - p_n)^2} + \sum_{n=1}^{m} \frac{c_n}{(x - p_n)^3} + \ldots$$

where the coefficients a_n, b_n, c_n, etc. are given by quite complicated expressions in terms of the polynomials P and Q and their derivatives, evaluted at $x = p_n$. Limiting our discussion to double roots, we find

$$a_n = \frac{Q(p_n)}{P'(p_n)} \qquad \text{for } \ell + 1 \le n \le K$$

$$\left.\begin{aligned} a_n &= \frac{2Q'(p_n)}{P''(p_n)} - \frac{2}{3}\,\frac{P'''(p_n)}{[P''(p_n)]^2}\,Q(p_n) \\ b_n &= \frac{2Q(p_n)}{P''(p_n)} \end{aligned}\right\} \qquad \text{for } 1 \le n \le \ell$$

In practice it is simpler to calculate the coefficients by defining the polynomials $R_n(x) = \frac{Q(x)}{(x - x_n)}$, $S_n(x) = \frac{Q(x)}{(x - x_n)^2}$. Then

$$a_n = \frac{Q(p_n)}{R_n(p_n)} \qquad \text{for } \ell + 1 \le n \le K$$

$$\left.\begin{aligned} a_n &= \left[\frac{Q(x)}{S_n(x)}\right]'_{x=p_n} \\ b_n &= \frac{Q(p_n)}{S_n(p_n)} \end{aligned}\right\} \qquad \text{for } 1 \le n \le \ell$$

With the help of the sum formulae (see Tables I and II)

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n - a)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

$$\sum_{n=-\infty}^{\infty} \frac{\cos n\theta}{(n - a)^2} = \pi^2 \left[\frac{\cos \theta a}{\sin^2 \pi a} - \frac{\theta}{\pi}\,\frac{\sin a(\pi-\theta)}{\sin \pi a}\right]$$

$$\sum_{n=-\infty}^{\infty} \frac{\sin n\theta}{(n - a)^2} = \pi^2 \left[\frac{\sin \theta a}{\sin^2 \pi a} - \frac{\theta}{\pi}\,\frac{\cos a(\pi-\theta)}{\sin \pi a}\right],$$

we obtain thus

$$\sum_{n=-\infty}^{\infty} \frac{Q(n)}{P(n)} = -\pi \sum_{n=1}^{K} a_n \cot \pi p_n + \pi^2 \sum_{n=1}^{\ell} \frac{b_n}{\sin^2 \pi p_n}$$

$$\sum_{n=-\infty}^{\infty} \frac{Q(n)}{P(n)} \cos n\theta = -\pi \sum_{n=1}^{K} a_n \frac{\cos p_n(\pi-\theta)}{\sin \pi p_n} + \pi^2 \sum_{n=1}^{\ell} b_n \left[\frac{\cos \theta p_n}{\sin^2 \pi p_n} - \frac{\theta}{\pi} \frac{\sin p_n(\pi-\theta)}{\sin \pi p_n}\right]$$

$$\sum_{n=-\infty}^{\infty} \frac{Q(n)}{P(n)} \sin n\theta = -\pi \sum_{n=1}^{K} a_n \frac{\sin p_n(\pi-\theta)}{\sin \pi p_n} + \pi^2 \sum_{n=1}^{\ell} b_n \left[\frac{\sin \theta p_n}{\sin^2 \pi p_n} - \frac{\theta}{\pi} \frac{\cos p_n(\pi-\theta)}{\sin \pi p_n}\right]$$

Extension to multiple roots of higher order is obvious. However, as the determination of the partial fraction coefficients becomes more complicated it is then often easier to take the simple expressions for distinct roots by adding different small quantities to the multiple roots, and then to take these small quantities to zero by a limiting process.