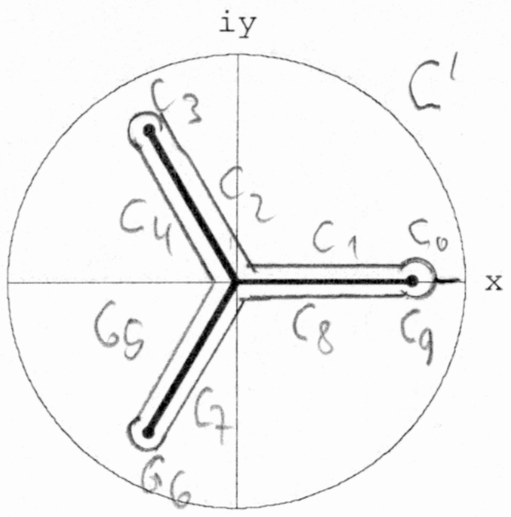
13.8.1.5 Werteverteilung von und Integral über eine Kubikwurzel.

$$J=\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}}=\frac{2\pi}{3\sqrt{3}} \quad ; \quad f(z)=\frac{1}{\sqrt[3]{z^3-1}} \,.$$



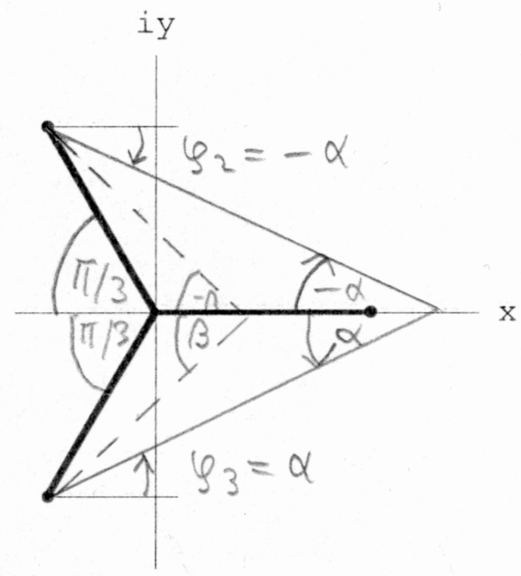
$$z^3-1=(z-z_1)(z-z_2)(z-z_3)=:$$
$$=r_1e^{i\varphi_1}\cdot r_2e^{i\varphi_2}\cdot r_3e^{i\varphi_3}\,.$$
$$\sqrt[3]{z^3-1}=\sqrt{r_1r_2r_3}\;e^{\frac{i}{3}\phi}\,e^{i2\pi k/3}\,,\quad k=0,1,2$$
$$=\sqrt{r^3-1} \quad \text{für } r=x>1 \Rightarrow k=0$$

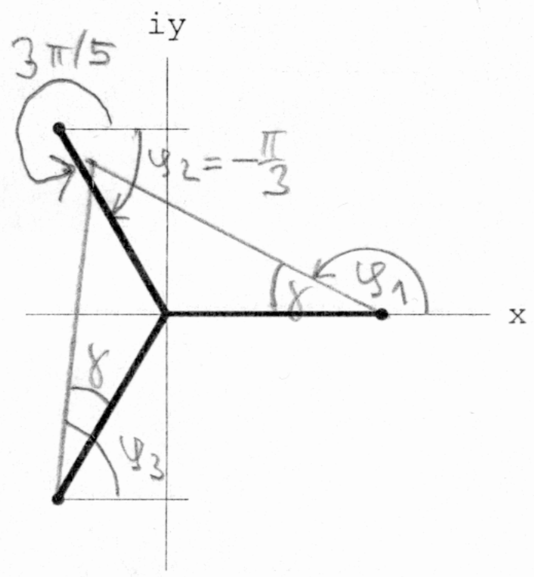
$z_1=e^{i2\pi/3}$, $z_3=1$.
$z_2=e^{i4\pi/3}=e^{-i2\pi/3}$.
$\phi:=\varphi_1+\varphi_2+\varphi_3$,

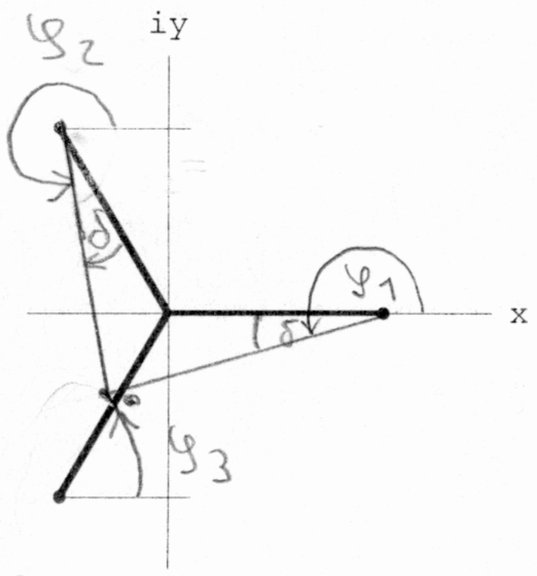
$$\oint_C f(z)dz=\oint_{C'} f(z)dz \qquad C=\sum_{i=0}^{9} C_i$$

$$f(z)=\frac{e^{-i\phi_3}}{\sqrt[3]{1-r^3}}$$

	φ_1	φ_2	φ_3	ϕ	$-\phi/3$	dz
$z=x=r>1$	0	$-\alpha$	α	0	0	
C_0	$0\to\pi$	$0\to 0$	$0\to 0$	π	$-\pi/3$	$\varepsilon e^{-i\phi/3}d\varphi_1 \xrightarrow{\varepsilon\to 0} 0$
C_1	π	$-\beta$	β	π	$-\pi/3$	dr
C_2	$\pi-\delta$	$-\frac{\pi}{3}$	$\frac{\pi}{3}+\delta$	π	$-\pi/3$	$dre^{i2\pi/3}$
C_3	$\pi-\delta\to\pi-\gamma$	$-\frac{\pi}{3}\to\frac{3\pi}{5}$	$\frac{\pi}{3}+\delta\to\frac{\pi}{3}+\gamma$			$\varepsilon e^{-i\phi/3}d\varphi_2 \xrightarrow{\varepsilon\to 0} 0$
C_4	$\pi-\delta$	$\frac{5\pi}{3}$	$\frac{\pi}{3}+\delta$	3π	$-\pi$	$dre^{i2\pi/3}$
C_5	$\pi-\delta$	$\frac{5\pi}{3}+\delta$	$\frac{\pi}{3}$	3π	$-\pi$	$dre^{-2\pi i/3}$
C_6	$\pi-\delta\to\pi-\delta$	$\frac{5\pi}{3}+\delta\to\frac{5\pi}{3}+\delta$	$\frac{\pi}{3}\to\frac{7\pi}{3}$			$\varepsilon e^{-i\phi/3}d\varphi_3 \xrightarrow{\varepsilon\to 0} 0$
C_7	$\pi-\delta$	$\frac{5\pi}{3}+\delta$	$\frac{7\pi}{3}$	5π	$-\frac{5\pi}{3}$	$dre^{-2\pi i/3}$
C_8	π	$\frac{5\pi}{3}+\vartheta$	$\frac{7\pi}{3}-\vartheta$	5π	$-\frac{5\pi}{3}$	dr
C_9	$\pi\to 2\pi$	$\frac{5\pi}{3}+\vartheta\to\frac{5\pi}{3}$	$\frac{7\pi}{3}-\vartheta\to\frac{6\pi}{3}$			$\varepsilon e^{i\phi/3}d\varphi_4 \xrightarrow{\varepsilon\to 0} 0$







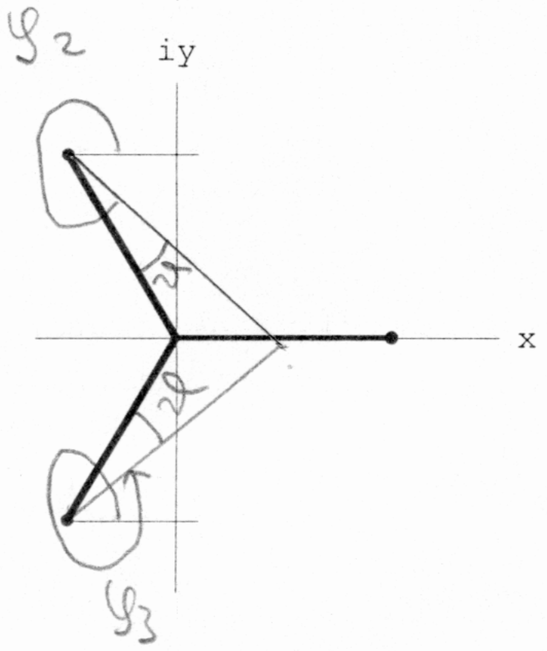


Abb. 13.10

$$= J\left(-e^{-i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}e^{i\frac{2\pi}{3}}-e^{-i\pi}e^{i\frac{2\pi}{3}}+e^{-i\pi}e^{-\frac{2\pi i}{3}}-e^{-\frac{5\pi i}{3}}e^{-\frac{2\pi i}{3}}+e^{-\frac{5\pi i}{3}}\right)$$
$$= J\left(-e^{-i\pi/3}+e^{i\pi/3}-e^{-i\pi/3}+e^{-i5\pi/3}-e^{-i7\pi/3}+e^{i\pi/3}\right)$$
$$= \underline{J\,6i\sin\tfrac{\pi}{3}} = \oint_{C'=|z|=R} f(z)dz = \oint \frac{dz}{z^2}\frac{1}{\sqrt[3]{\frac{1}{z^3}-1}} = \oint \frac{dz}{z}\frac{1}{\sqrt{1-z^3}} = \underline{2\pi i\cdot 1}$$