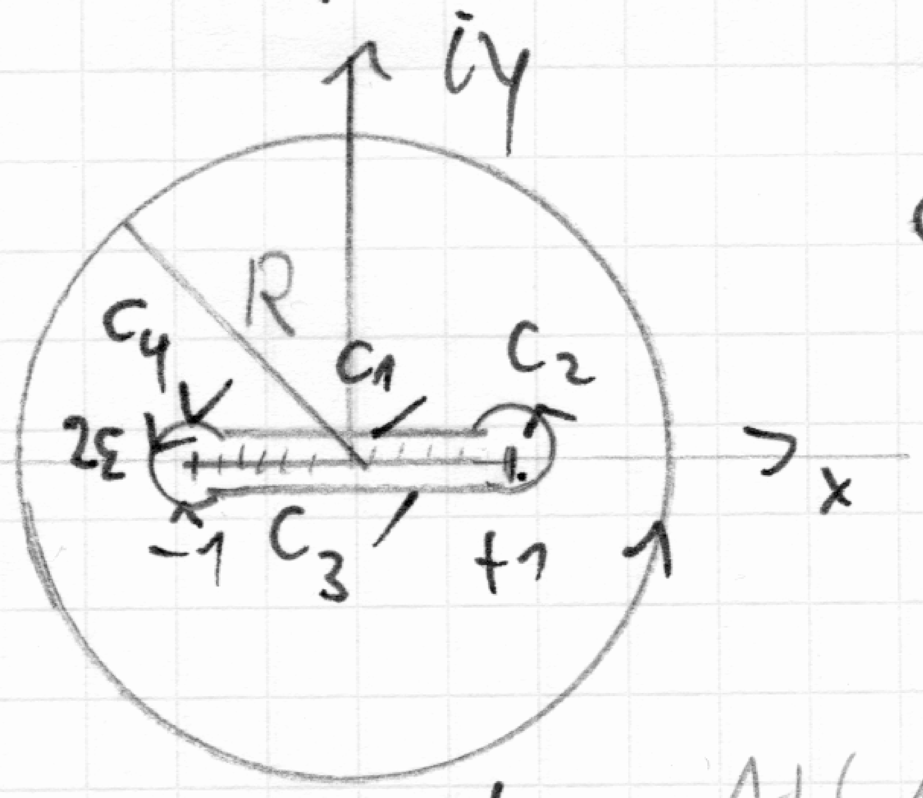
13.8.1.4 (Fts).

Eine Anwendung ist die Berechnung des folgenden Integrals:

$$I = \int_{-1}^{1} x^{2n}\sqrt{1-x^2}\,dx = 2\int_0^1 x^{2n}\sqrt{1-x^2}\,dx = \pi(-1)^n\binom{1/2}{n+1}, \quad n\in\mathbb{N}_0.$$
$$= \pi\,(2n-1)!!\,/\,[2^{n+1}\,(n+1)!]$$



C' Abb 139

$$\oint_{C=C_1+C_2+C_3+C_4} f(z)\,dz = \oint_{C':\,|z|=R} f(z)\,dz = \oint_{1/C} f\left(\frac{1}{\varphi}\right)\frac{d\varphi}{\varphi^2}, \quad f(z) = z^{2n}\sqrt{z^2-1}$$

C_1: $z = r,\ 1 \geq r \geq -1,\ f(z) = \sqrt{1-r^2}\,e^{i\pi/2}$

C_3: $z = r,\ -1 \leq r \leq 1,\ f(z) = \sqrt{1-r^2}\,e^{-i\pi/2}$

C_2: $z = +1 + \varepsilon e^{i\varphi},\ -\pi \leq \varphi \leq \pi;\quad |I_2| \leq \left|\int_{-\pi}^{\pi} d\varphi\,\varepsilon e^{i\varphi}\frac{\sqrt{1-\varepsilon^2 e^{2i\varphi}}}{1+\varepsilon^2 e^{2i\varphi}}\right| \xrightarrow[\varepsilon\to 0]{} 0$

C_4: $z = -1 + \varepsilon e^{i\varphi},\ 0 \leq \varphi \leq 2\pi;\quad |I_4| \leq \int_0^{2\pi} d\varphi\,\varepsilon e^{i\varphi}\frac{\sqrt{-1+\varepsilon^2 e^{2i\varphi}}}{1}$

$$\oint_C f(z)\,dz = \underbrace{\int_{1-\varepsilon}^{-1+\varepsilon} r^{2n}dr\sqrt{1-r^2}\,\underbrace{e^{i\pi/2}}_{i}}_{C_1} + \underbrace{\int_{-1+\varepsilon}^{1-\varepsilon} dr\sqrt{1-r^2}\,\underbrace{e^{-i\pi/2}}_{-i}\,r^{2n}}_{C_3} + \underbrace{I_2 + I_4}_{\xrightarrow[\varepsilon\to 0]{} 0}$$

$$\stackrel{\varepsilon\to 0}{=} -2i\int_{-1}^{1} dx\,x^{2n}\sqrt{1-x^2} = \oint_{C'} f(z)\,dz$$

nach Monodromiesatz

$$z = \frac{1}{\varphi},\quad dz = -\frac{d\varphi}{\varphi^2}$$

$C': Re^{i\varphi} = z,\ 1/C': \varphi = \frac{1}{R}e^{-i\varphi}$

$|\varphi| = \frac{1}{R} < 1$

$$= \oint_{1/C'} \frac{1}{\varphi^{2+2n}}\sqrt{\frac{1}{\varphi^2}-1}\,d\varphi$$

$$\underline{\underline{= -2i\int_{-1}^{1} x^{2n}dx\sqrt{1-x^2}}} = \oint_{1/C'} \frac{1}{\varphi^{3+2n}}\sqrt{1-\varphi^2}\,d\varphi = 2\pi i\,\mathrm{Res}\left(\frac{\sqrt{1-\varphi^2}}{\varphi^{3+2n}},\ \varphi=0\right) \underline{\underline{= -2\pi i(-1)^n\binom{1/2}{n+1}}}$$

$\varphi = 0$: Pol, Ordg $2n+3$.

$$\sqrt{1-\varphi^2} = (1-\varphi^2)^{1/2} = 1 + \sum_{k=1}^{\infty}\binom{1/2}{k}(-\varphi^2)^k, \quad \frac{\sqrt{1-\varphi^2}}{\varphi^{3+2n}} = \frac{1}{\varphi^{3+2n}} + \sum_{n\geq 0,\,k=1}^{\infty}\binom{1/2}{k}(-1)^k\varphi^{2k-3-2n}$$

$$\mathrm{Res}\left(\frac{\sqrt{1-\varphi^2}}{\varphi^{3+2n}},\ \varphi=0\right) = -(-1)^n\binom{1/2}{n+1}. \quad \Big|\ 2k-3-2n = -1 \to \text{Residuum},\quad k = n+1$$

$$\binom{1/2}{n+1} = \frac{1/2\cdot(-1/2)\cdots(1/2-n)}{(n+1)!} = \frac{(-)^n\,(2n-1)!!}{2^{n+1}\,(n+1)!}$$

```
Integrate[x^k Sqrt[1 - x^2], {x, -1, 1}, Assumptions → Re[k] > -1]
```

$$\frac{(1+(-1)^k)\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{1+k}{2}\right]}{4\,\mathrm{Gamma}\left[2+\frac{k}{2}\right]}$$

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Integrate[x^k Sqrt[1 - x^2], {x, 0, 1}, Assumptions → Re[k] > -1]
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$$\frac{\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{1+k}{2}\right]}{4\,\mathrm{Gamma}\left[2+\frac{k}{2}\right]}$$