3.8.1.3 (Fts.)

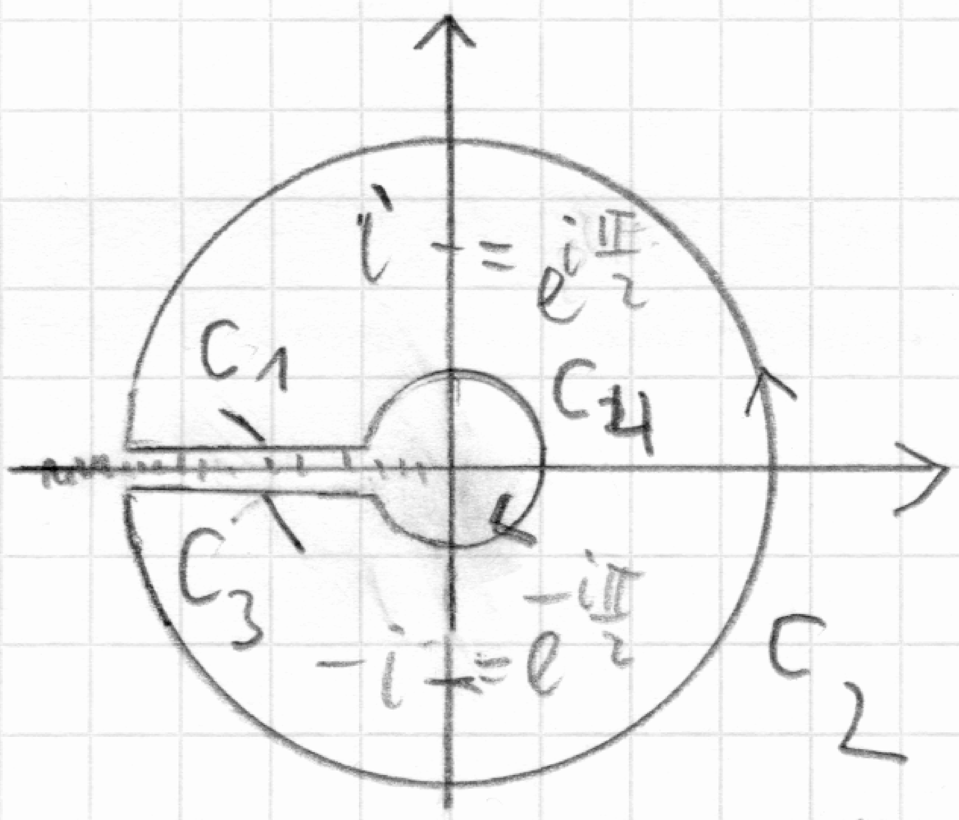

Abb.13.7

$$\oint f(z)\,dz = 2\pi i\left[\mathrm{Res}(f(z), z=i) + \mathrm{Res}(f(z), z=-i)\right]$$

Imaginäre e-Potenzen gemäß Zweig

$$f(z) = \frac{z^{\beta-1}}{z^2+1}, \quad 0<\beta<2$$

$$\mathrm{Res}\left(\frac{z^{\beta-1}}{(z+i)(z-i)}, z=\underset{=i}{e^{\frac{i\pi}{2}}}\right) = \frac{e^{i\frac{\pi}{2}(\beta-1)}}{2i}; \quad \mathrm{Res}\left(f(z), z=\underset{=-i}{e^{-i\pi/2}}\right) = \frac{e^{-i\frac{\pi}{2}(\beta-1)}}{-2i}$$

C_1: $R \geq r \geq \varepsilon$, $\varphi=\pi$; $z^{\beta-1} = r^{\beta-1} e^{i\pi(\beta-1)}$

C_2: $z = Re^{i\varphi}$, $-\pi \leq \varphi \leq \pi$; $z^{\beta-1} = R^{\beta-1} e^{i\varphi(\beta-1)}$ $\quad |I_2| \leq \left|\int_{-\pi}^{\pi} d\varphi \frac{R^{\beta-1} R}{1+R^2 e^{2i\varphi}}\right| = O(R^{\beta-2}) \underset{R\to\infty}{\longrightarrow} 0$

C_3: $\varepsilon \leq r \leq R$, $\varphi = -\pi$; $z^{\beta-1} = r^{\beta-1} e^{-i\pi(\beta-1)}$

C_4: $z = \varepsilon e^{i\varphi}$, $\pi \geq \varphi \geq -\pi$; $z^{\beta-1} = \varepsilon^{\beta-1} e^{i\varphi(\beta-1)}$ $\quad |I_4| \leq \left|\int_{\pi}^{-\pi} d\varphi \frac{\varepsilon^{\beta}}{1+\varepsilon^2 e^{2i\varphi}}\right| = O(\varepsilon^{\beta}) \underset{\varepsilon\to 0}{\longrightarrow} 0$

$$e^{i(\beta-1)\pi}\int_{\infty}^{0} \frac{r^{\beta-1}dr}{1+r^2} e^{i\pi} + e^{-i(\beta-1)\pi}\int_0^{\infty} \frac{r^{\beta-1}dr}{1+r^2} e^{-i\pi} = \frac{2\pi i}{2i}\left[e^{i(\beta-1)\frac{\pi}{2}} - e^{-i(\beta-1)\frac{\pi}{2}}\right]$$
$$= -\pi 2i \cos(\beta\pi/2)$$

$$\int_0^{\infty} \frac{x^{\beta-1}dx}{1+x^2}\left(e^{-i\beta\pi} - e^{i\beta\pi}\right) = \underset{2\sin(\frac{\beta\pi}{2})\cos(\frac{\beta\pi}{2})}{-2i\sin(\beta\pi)}\int_0^{\infty}\frac{x^{\beta-1}dx}{1+x^2} = -2i\pi\cos(\beta\pi/2)$$

3.8.1.4 $\quad f(z) = \sqrt{z^2-1}$

$\sqrt{z^2-1} = \sqrt{z+1}\sqrt{z-1}$ $\quad$ Verzweigungspunkte: $z = 1, -1$.

$z - z_1 = r_1 e^{i\varphi_1} = z-1$. $\quad \sqrt{z^2-1} = \sqrt{r_1 e^{i\varphi_1}}\sqrt{r_2 e^{i\varphi_2}}\sqrt{e^{2\pi i K}}$

$z - z_2 = r_2 e^{i\varphi_2} = z+1$. $\quad = \sqrt{r_1 r_2}\, e^{i\varphi_1/2} e^{i\varphi_2/2} e^{\pi i K}, K = 0, 1.$

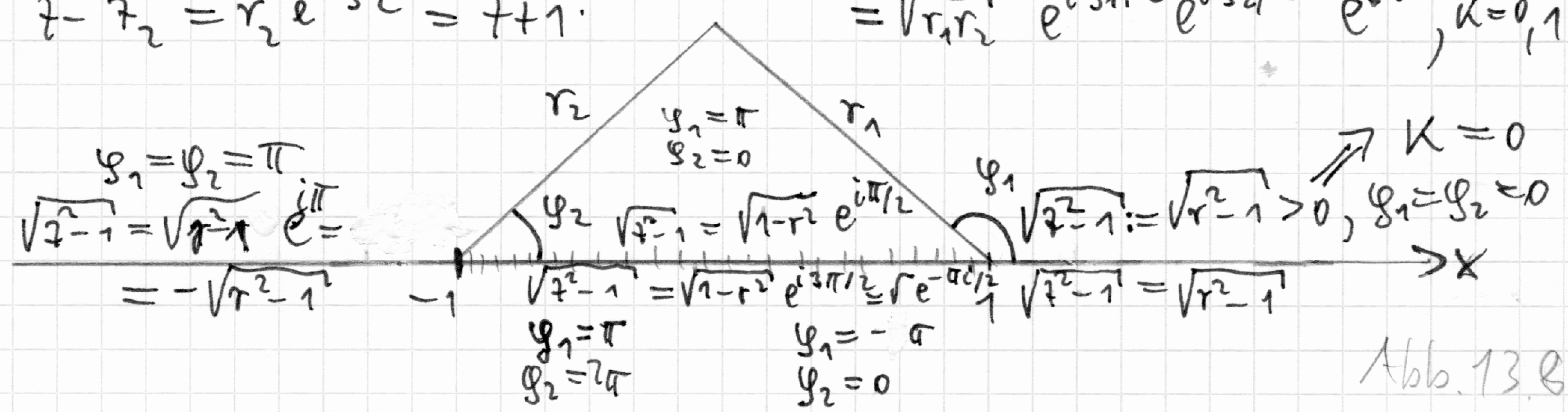

Abb.13.8

$|z| \gg 1$: $\sqrt{z^2-1} = z\sqrt{1-\frac{1}{z^2}} = z\left(1 - \frac{1}{2z^2} - \frac{1}{8z^4} \cdots\right)$

$z = r \gg 1 \quad \sqrt{z^2-1} \approx r \qquad z = ir \gg i \quad \varphi_1 = \frac{\pi}{2} + \alpha, \varphi_2 = \frac{\pi}{2} - \alpha$

$z = -r \ll -1 \quad \sqrt{z^2-1} \approx -r \qquad \sqrt{z^2-1} \approx \underset{r^2}{\sqrt{r_1 r_2}}\, e^{i\varphi_1} e^{i\varphi_2} = r\,e^{i\pi/2}$

$z = -ir \ll -i$, $\sqrt{z^2-1} \approx r \ldots\ldots$