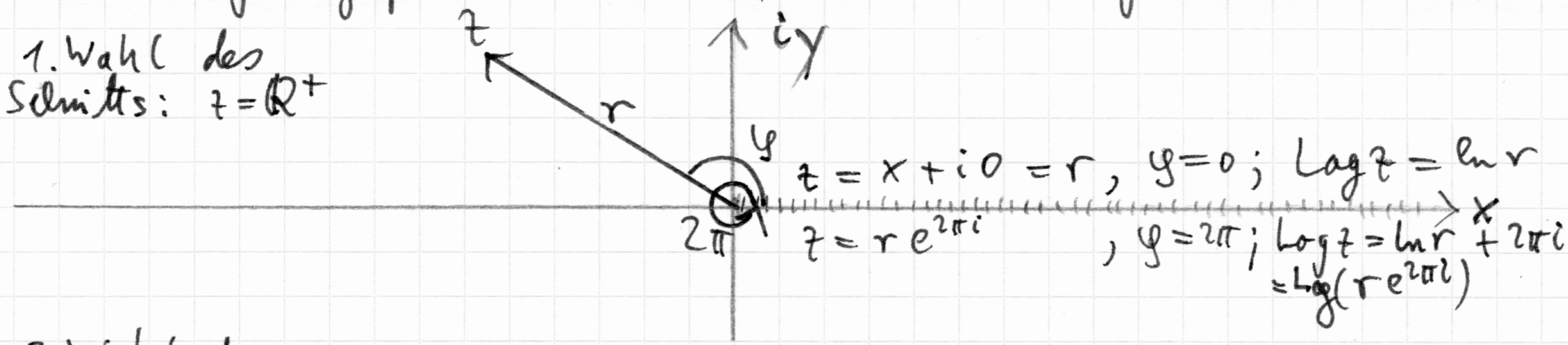
3.8.1.1. Log(z)

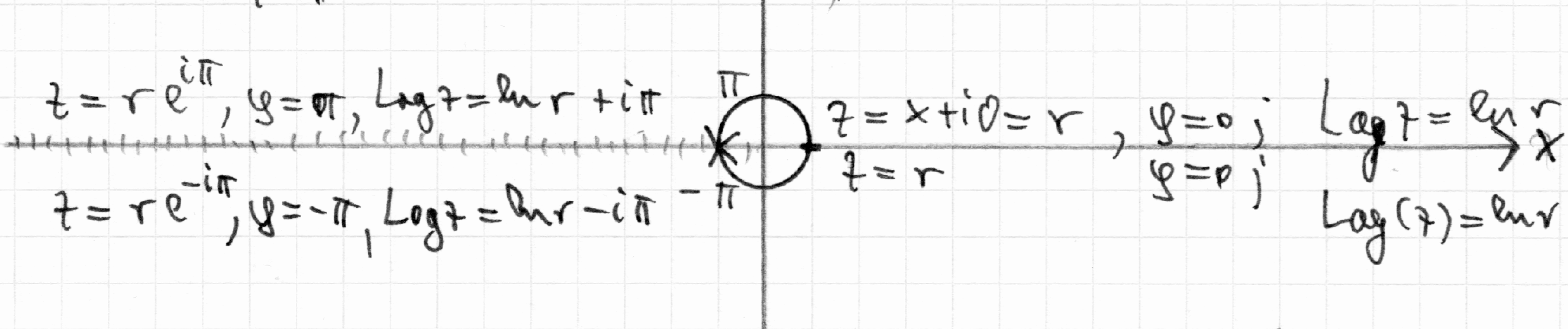
$$\mathrm{Log}(z) = \ln|z| + i\arg(z) + 2\pi i K, \quad K \in \mathbb{Z}.$$

Verzweigungspunkte: $z = 0, \infty$. $\qquad \log(z) = \ln r + i\varphi$

1. Wahl des Schnitts: $z = \mathbb{R}^+$



2. Wahl des Schnitts: $z = \mathbb{R}^-$



Berechnung des Integrals:

$$I = \int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0 \qquad \oint f(z)\,dz,\ f(z) = \frac{\mathrm{Log}\,z}{1+z^2}$$

Weg C_i für Integral I_i.

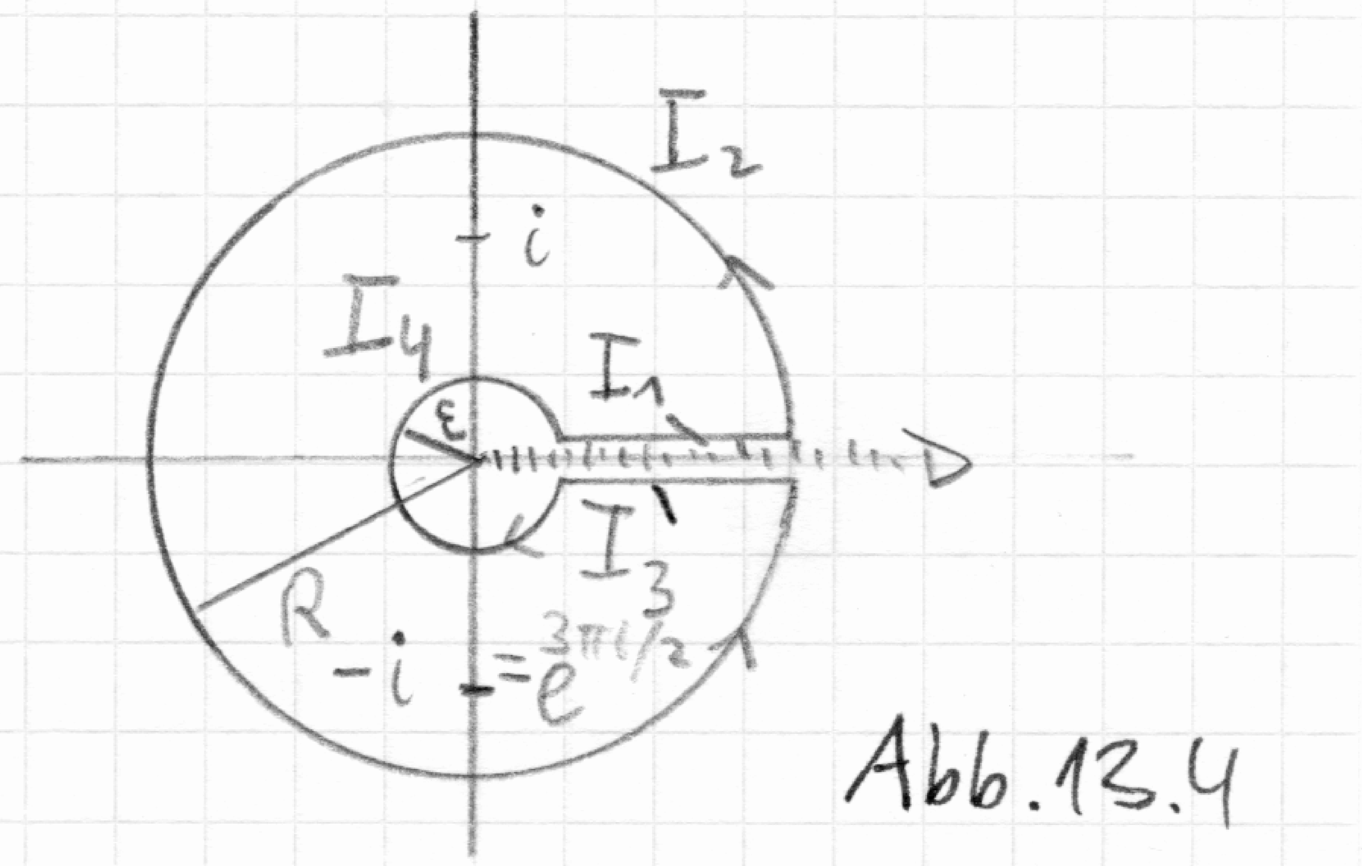


Abb. 13.4

C_1: $\varepsilon \le x \le R,\ \varphi = 0,\ \mathrm{Log}\,z = \ln x$

C_2: $z = R e^{i\varphi},\ 0 \le \varphi \le 2\pi,\ \mathrm{Log}(z) = \mathrm{Log}(R e^{i\varphi}) = \mathrm{Log}\,R + i\varphi$

C_3: $z = r e^{i2\pi},\ R \ge r \ge \varepsilon,\ \mathrm{Log}(z) = \mathrm{Log}(r e^{2\pi i}) = \ln r + 2\pi i$

C_4: $z = \varepsilon e^{i\varphi},\ 2\pi \ge \varphi \ge 0,\ \mathrm{Log}(z) = \ln\varepsilon + i\varphi$

$$I_2 = \int_0^{2\pi} \frac{(\ln R + i\varphi)}{1 + R^2 e^{2i\varphi}}\, iR e^{i\varphi}\,d\varphi, \quad |I_2| \le \int_0^{2\pi} d\varphi\, R\, \frac{\ln R + 2\pi}{R^2 - 1} \xrightarrow[R\to\infty]{} 0$$

$$I_4 = \int_{2\pi}^0 \frac{\ln\varepsilon + i\varphi}{1 + \varepsilon^2 e^{2i\varphi}}\, i\varepsilon e^{i\varphi}\,d\varphi, \quad |I_4| \le \int_{2\pi}^0 d\varphi\, \varepsilon\, \frac{\ln\varepsilon + 2\pi}{1+\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} 0$$

$$\mathrm{Res}\left(\frac{\mathrm{Log}(z)}{(z+i)(z-i)},\ z = e^{i\pi/2} = i\right) = \frac{\mathrm{Log}(e^{i\pi/2})}{2i} = \frac{\pi}{4}, \quad \mathrm{Res}\left(\frac{\mathrm{Log}(z)}{(z+i)(z-i)},\ z = e^{3\pi i/2} = -i\right) = \frac{\mathrm{Log}(e^{3\pi i/2})}{-2i} = -\frac{3\pi}{4}$$

$$I_1 + I_3 = \int_0^\infty \frac{dr\,\ln r}{1+r^2} + \int_\infty^0 \frac{\ln r + 2\pi i}{1+r^2}\,dr = -2\pi i \int_0^\infty \frac{dr}{1+r^2} = 2\pi i\left(\frac{\pi}{4} - \frac{3\pi}{4}\right) = 2\pi i\left(-\frac{\pi}{2}\right)$$

$$f(z) = \frac{(\mathrm{Log}\,z)^2}{1+z^2}, \quad I_1 + I_3 = \cancel{\int_0^\infty \frac{(\ln r)^2}{1+r^2}\,dr} - \int_0^\infty \frac{dr}{1+r^2}\left[\cancel{(\ln r)^2} + 4\pi i \ln r - 4\pi^2\right]$$

$$= 4\pi i \int_0^\infty \frac{\ln r}{1+r^2}\,dr - 4\pi^2 \int_0^\infty \frac{dr}{1+r^2} = \cancel{2}\pi i \underbrace{\left[\frac{(\ln 1 + i\pi/2)^2}{\cancel{2i}} + \frac{(\ln 1 + i3\pi/2)^2}{-\cancel{2i}}\right]}_{\text{reell}}$$

$$I = 0$$