Chapter 10

The δ-Distribution

10.1 Heuristic considerations

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10.2 Mathematical foundations of the δ-distribution

The considerations of the preceeding section are of a purely heuristic nature. In particular, the requirements in eqs.(10.7) and (10.8) prescribed for the "δ-distribution" are incompatible with the common definition of a function as used in mathematics. The "δ-Funktion" ist no function, it is a **Distribution** (or **generalized function**): The limit of a sequence of functions under an integral. Each function of the sequence depends on x, $s_n(x) \rightarrow s(x)$, so does the limit function $s(x)$. However, the limit of the sequence of functions may not be a function, it may be a distribution.

One encounters a similar case in the theory of numbers: An infinite sequence of rational numbers may converge towards a number, which is no longer a fraction, but an irrational number. For example, from the binomial series we find for $x = 1$:

$$\begin{aligned}
(1+x)^{1/2} &= \sum_{n=0}^{\infty} \begin{pmatrix} 1/2 \\ n \end{pmatrix} x^n, \quad \hat{=} \quad s_0(x), s_1(x), \dots, s_n(x) \rightarrow s(x) \\
\sqrt{2} &= \sum_{n=0}^{\infty} \begin{pmatrix} 1/2 \\ n \end{pmatrix} \quad \hat{=} \quad s_0, s_1, \dots, s_n \rightarrow s = \sqrt{2} \\
s_n &= \sum_{k=0}^{n} \begin{pmatrix} 1/2 \\ n \end{pmatrix} \quad \hat{=} \quad s_0 = 1,\ s_1 = \frac{3}{2},\ s_2 = \frac{11}{8},\ s_3 = \frac{23}{16}, \dots,\ s_9 = \frac{93009}{65536}, \dots
\end{aligned}$$

For numbers see notebook K10SequSqrt2.nb .

A similar case is the Fourier series, Bs.2 on p.6.3. Each term is a continuous function. A finite sum of such terms is a finite function. The limit function is discontinuous.

A rigorous theory of distributions cannot cope with the two conditions eqs.(10.7) and (10.8) but only with the following prescription:

$$\int_L \delta(x - x')\ F(x)\ dx = \begin{cases} F(x') & \text{f'ur} \quad x' \in L, \\ 0 & \text{f'ur} \quad x' \notin L. \end{cases} \tag{10.1}$$

A graphic representation of this requirement may be interpreted as follows: The δ-Distribution has a sharp spike, which selects the value of the function $F(x)$ at the point $x = x'$, i.e. $F(x')$;

it is zero anywhere else. It is obvious that such an approach works only, if the function $F(x)$ is continuous at the point $x = x'$. More properties will be prescriped to this function below.

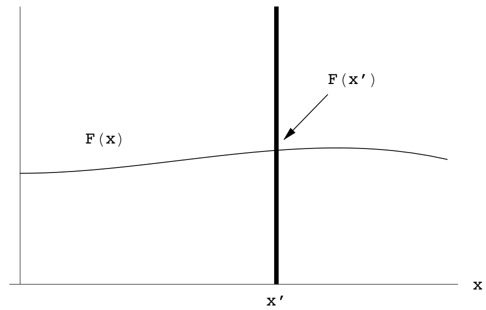

Figure 10.1: The δ-distribution extracts the value $F(x')$ from all the values the function $F(x)$ may assume.

The relation introduced above is called a linear functional; it suffices completely for a rigorous mathematical derivation of the results, the physicists obtain with the help of the much less rigorous and vague notion "δ-function".

Without loss of generality we set $x' = 0$ i the above definition:

$$\int_L \delta(x)\ F(x)\ dx \ = \ \begin{cases} F(0) & \text{f"ur} \quad 0 \in L, \\ 0 & \text{f"ur} \quad 0 \notin L. \end{cases} \tag{10.2}$$

An exact foundation will be given for this equation. In doing this we follow Lighthill, (Chap.2 and parts of 1). At first we must list and explain some basic definitions.

10.2.1 Definitions of some basic terms

Def.7.1: $f(x)$ is called a **test function** (G.: Grundfunktion, if it is arbitrarily often differentiable C^∞ everywhere in $-\infty \ \leq \ x \ \leq \ \infty$ and if it together with all its derivatives tends to zero faster than any power at infinity; i.e. it fulfils the following limit:

$$\lim_{|x| \to \infty} \ f^{(k)}(x) \ = \ O(|x|^{-N}) \tag{10.3}$$

with arbitrary N.
The symbol $0(g)$ means: it is an expression of the order g at the largest, or

$$f \ = \ O(g) \quad \Leftrightarrow \quad |f| \ < \ A|g|$$

for a suitable, finite constant A. An example of a test function is: e^{-x^2}.

Def.7.2: $F(x)$ is called **slowly increasing** (G.: schwach wachsend), if all derivatives of $F(x)$ exist and if $F(x)$ together with all its derivatives diverges at infinity not stronger than a power; so it fulfils the following limit:

$$\lim_{|x| \to \infty} \ F(x) \ = \ O(|x|^{N}) \tag{10.4}$$

for a suitable N. An example of a slowly growing function is any polynomial.

The derivative of a test function is a test function. The sum and the difference of two test functions are test functions. The product of two test functions, or that of a test function and a slowly growing function are test functions.

The δ-distribution is defined with the help of the following sequence of functions: (cf.fig.10.2)

$$\delta_n(x) \;:=\; \sqrt{\frac{n}{\pi}}\; e^{-nx^2}\;, \quad \Rightarrow \quad \int_{-\infty}^{\infty} \delta_n(x)\; dx \;=\; 1. \tag{10.5}$$

Each $\delta_n(x)$ is a Gaussian. The area below any of these curves has the value 1. The Gaussian is the narrower and the higher, the larger n. This gives a spike in the limit $n \to \infty$. However, it is not allowed to write:

$$\lim_{n\to\infty}\; \delta_n(x) \;=\; \delta(x).$$

But we get for any arbitrary test function $F(x)$:

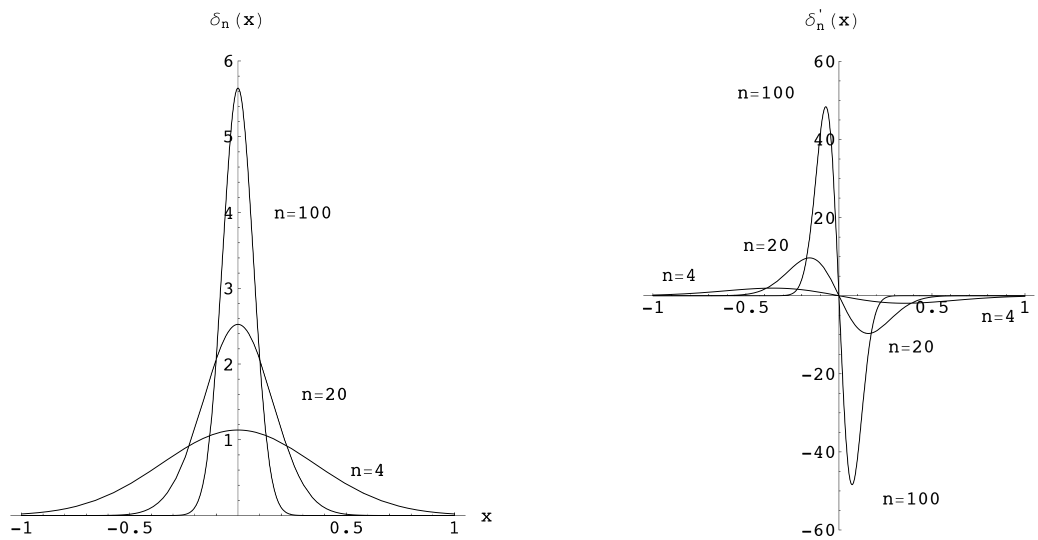

Figure 10.2: Left: Graphs of the functions $\delta_n(x)$ forr $n = 4, 20, 100$. These converge towards a point source located at the point $x = 0$. Right: The derivatives of these functions. They converge towards a dipole source located at the point $x = 0$.

$$\lim_{n\to\infty}\; \int_{-\infty}^{\infty} \delta_n(x)\; F(x) \;=\; F(0). \tag{10.6}$$

and this is equivalent to (10.2).

For a proof of eq.(10.6) we consider:

$$\mathcal{I} \;=\; \left| \int_{-\infty}^{\infty} e^{-nx^2}\; \sqrt{\frac{n}{\pi}}\; F(x)\; dx \;-\; F(0) \right| \;=\; \left| \int_{-\infty}^{\infty} e^{-nx^2}\; \sqrt{\frac{n}{\pi}}\; [F(x) \;-\; F(0)]\; dx \right|.$$

According to the mean value theorem of the differential calculus we have:

$$F(x) \;-\; F(0) \;=\; x\; F'(\theta x) \quad \text{mit} \quad 0 \;\leq \theta \;\leq 1;$$

Inserting this into the preceeding equation gives:ein, ergibt sich:

$$\mathcal{I} \;\leq\; \max|F'(x)|\; \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nx^2}\; |x|\; dx \;=\; \frac{1}{\sqrt{\pi n}}\; \max|F'(x)| \;\longrightarrow\; 0 \quad \text{f"ur} \quad n \to \infty.$$

There is not only the sequence of functions defined in eq. (10.5) but a set of sequences, whose limits give equivalent representations of the δ-distribution. For example, from

$$\int_{-\infty}^{\infty} e^{-n^\nu x^{2\nu}}\; dx \;=\; \Gamma(1/2\nu)/\nu\sqrt{n},$$

if follows for every real ν that the sequences

$$\delta_n(x) \;=\; \frac{\sqrt{n}\;\nu}{\Gamma(1/2\nu)}\; e^{-n^\nu x^{2\nu}}$$

represent the δ-distribution.

The definite integral of the δ-distribution is the Heavisidesche unit step function:

$$\theta(x) \;=\; \int_{-\infty}^{x} \delta(\bar{x})\; d\bar{x} \;=\; \begin{cases} 0 & \text{for} \quad -\infty < x < 0, \\ \frac{1}{2} & \text{for} \quad x = 0, \\ 1 & \text{for} \quad 0 < x < \infty \end{cases}$$

10.2.2 The Heaviside unit step function

By integrating eq.(10.5) we get for $\theta_n(x)$ (s.Fig.10.3):

$$\theta_n(x) \;=\; \int_{-\infty}^{x} \delta_n(\bar{x})\; d\bar{x} \;=\; \frac{1}{2}\Big[1 \;+\; \mathrm{erf}(\sqrt{n}\;x)\Big].$$

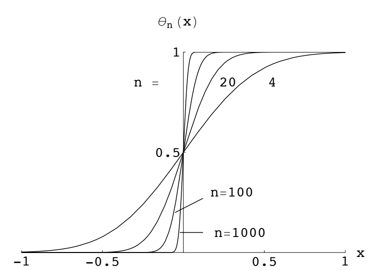

Figure 10.3: The sequence of functions $\theta_n(x)$ for $n = 4, 20, 100, 1000$.

10.3 The completeness relation