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# Bell's Theorem
## A Review of Theory and Experiment

v1.0

Bachelor's Thesis

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# Abstract

Bell's Theorem took the discussion around the fundamentals of Quantum Mechanics from thought experiments back to experimental evidence. Einstein, Podolsky and Rosen (EPR) proposed a thought experiment to show that quantum mechanics is incomplete. Based on the assumptions used by EPR, Bell derived an inequality which showed that these assumptions, *locality* and *reality*, are incompatible with Quantum Mechanics. This inequality was improved to allow experimental tests of the assumptions. The experiments showed that the assumptions made by EPR are not only incompatible with quantum mechanics but more importantly with nature. However, the earlier experiments required additional assumptions which a model satisfying the assumptions made by EPR could exploit to still satisfy this inequality, leading to so called loopholes. Recent (2015) technological as well as theoretical improvements could close those loopholes and allowed for so called 'loophole free' experiments, where all of the previous major loopholes are closed. This thesis gives an overview of the basics of entangled states and describes the thought experiment by EPR, the so called EPR-Paradoxon. Different inequalities and their improvements are discussed. An overview of the performed experiments, different sources of entangled states as well as the loopholes and proposed remedies is given.



# Zusammenfassung

Bell's Theorem konnte die Diskussion um die Quantenmechanik von Gedankenexperimenten wieder auf experimentelle Ergebnisse zurückbringen. Einstein, Podolsky und Rosen (EPR) schlugen ein Gedankenexperiment vor, das zeigen sollte, dass die Quantenmechanik nicht vollständig ist. Bell konnte auf Basis der getroffenen Annahmen eine Ungleichung ableiten, auf Basis derer später Experimente durchgeführt werden konnten. Dadurch konnte gezeigt werden, dass die Annahmen von EPR nicht nur mit der Quantenmechanik, sondern auch mit der realen Welt inkompatibel sind. Jedoch erfordern die Experimente gewisse zusätzliche Annahmen, die von einer Theorie ausgenutzt werden könnten, um sowohl die EPR-Annahmen zu erfüllen als auch die Ungleichung zu verletzen, was zu sogenannten loopholes führt. Neuere (2015) Experimente, die sogennanten loophole free experiments, konnten, aufgrund technologischer und theoretischer Verbesserungen diese loopholes schließen. Die vorliegende Arbeit gibt eine Übersicht über die Grundlagen zu verschränkten Zuständen, beschreibt das Gedankenexperiment von EPR (das sog. EPR-Paradoxon) und diskutiert verschiedene Ungleichungen und ihre Verbesserungen. Außerdem wird eine Übersicht über die bisher durchgeführten Experimente, einige der verwendeten Quellen von verschränkten Zuständen sowie der vorkommenden loopholes gegeben.

# Contents

# 1 Introduction

Around the year 1926, during the golden ages of Quantum Mechanics (QM), many physicists were dissatisfied with the theory because it can not predict the outcome of a single measurement but gives a probability distribution for the different possible outcomes of the measurement instead. In other words, they wanted a theory which could predict the outcome of a measurement *before* the measurement. This led to the idea that the theory is incomplete - if it would be completed, specific outcomes could be predicted. [56]

Contributing to this discussion in 1935, Einstein, Podolsky and Rosen (EPR) published the so called EPR paradox. It was intended show, based on what was later called an entangled state, that quantum mechanics is indeed an incomplete theory. In Schrödinger's response the famous cat made its first appearance [49, 47, 48]. This started a debate about the foundations of quantum mechanics, about determinism and causality, as the existence of these entangled states contradicted the classical understanding of the combination of predictability and locality [56].

In 1964, almost 30 years later, Bell formulated the first version of his famous theorem based on the EPR assumptions and was able to show that those assumptions are incompatible with quantum mechanics. Given the assumptions of locality and predictability he derived an inequality which is violated by the quantum mechanical prediction. Later, based on this work, other inequalities more suited for experimental purposes were derived and experimentally tested. The outcome of these experiments showed that the real world is incompatible with the combination of assumptions used by EPR to derive their paradox. This is especially important, as it brought the philosophical discussion back to experimental evidence. [10]

Since then many different experiments have been carried out with a variety of different systems (photons, electrons, KB-Mesons, atoms, etc.), most of them showing a violation of the inequality.

Still, the discussion around the foundations and completeness of QM continues. Different types of theories exist to complete QM. One such idea of completion is by hidden variables or hidden parameters - these variables would encode the measurement outcome and predefine it. If they were known, we could predict this outcome. More restrictive is the idea of local hidden variable (LHV), where the variable is localized in a sense that it can not get or transmit any information faster than the speed of light [41, 56]. Incorrect statements exist, that Bell's theorem eliminates all LHV theories or even contradicts hidden variable theories. Functioning hidden variable theories exist, such as Bohmian Mechanics. This theory was developed by de'Broglie and later Bohm. It supplements the wave function with additional coordinates, allowing for a remedy of the measurement problem [28].

Bells theorem is also an important piece in the foundation of Quantum Information Theory (QIT), as many applications of QIT build on entangled states. For example, in quantum cryptography, entangled states are distributed over a large distance in order to secure the exchange of encryption keys. To test if the distribution of the entangled state was successful, a Bell inequality can be employed. [57]

In the following we will briefly introduce concepts such as entangled states, the density operator, entanglement entropy and maximal entanglement which are vital to the understanding of the further chapters . Then the EPR paradox will be presented, as well as a simple contradiction of the EPR-assumptions to the QM predictions, based on multi particle entanglement.

In the second chapter, Bells original inequality and the underlying assumptions will be discussed. Two variants, the CHSH-inequality (after Clauser, Horne, Shimony and Holt [19]) and the CH-inequality will be introduced.

In the third chapter an overview of some of the experiments performed up to now shall be given. Experimental sources of entangled particles as well as possible loopholes in the experimental setups will be discussed leading up to the recently performed so called loophole free experiments.

Finally, the content of the most important topics will be summarized.

This text mainly follows Bertlmann and Zeilinger [10], Myrvold, Genovese, and Shimony [41], Evertz [23], Bell [8], Clauser et al. [19], Clauser and Horne [18], Zeilinger [58] and Giustina et al. [26].

## 1.1 Basics

A minimal introduction (following Evertz [23]) into the basic concepts and definitions necessary for the following derivations shall be given here. At first we will see a different representation of quantum mechanical states, called the 'density operator'. Then the concept of an entangled state will be shown and defined. Finally, in order to quantify entanglement, the concept of 'entanglement entropy' shall be introduced.

### 1.1.1 Entangled States

In a system with two (or more) degrees of freedom the state of the system can be a product state. Consider for example a system with polarized photons at different locations:

$$|\psi\rangle = |polarization\rangle \otimes |location\rangle \tag{1}$$

In this special case, when one is interested in only one of the degrees of freedom, the other can simply be omitted. However, this is not always the case. For many 3D-projections in cinemas, special glasses are used where e.g. the left eye is covered by a vertical polariser and the right eye is covered by a horizontal polariser. The resulting state would read

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|left, vertical\rangle + |right, horizontal\rangle). \tag{2}$$

Here it is impossible to factorize into the degrees of freedom as we have done above. A more general definition can be given for pure states in two degrees of freedom:

**Definition 1.1** (PURE ENTANGLED STATE)**.** *A pure state in two degrees of freedom (A and B) is entangled in a basis $|\alpha_n\rangle_A$, $|\beta_n\rangle_B$, if its basis representation is given by*

$$|\psi\rangle = \sum_n^N c_n |\alpha_n\rangle_A \otimes |\beta_n\rangle_B \,, \; N > 1 \tag{3}$$

*where $|\psi\rangle$ must not reduce to $N = 1$ with any separate basis transformation in A and B.*

Another common example of entangled states are spin states of two systems. Consider e.g. $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|{\downarrow_A}{\uparrow_B}\rangle_z - |{\uparrow_A}{\downarrow_B}\rangle_z)$. Here, if particle $A$ is measured to be in the state $|{\downarrow_A}\rangle_z$ a measurement of particle $B$ will find it in the state $|{\uparrow_B}\rangle_z$ and vice versa. Of course we do not have to measure the same state along the $z$ axis, but we could also choose to do so along the $x$, $y$ or any other axis. A basis transformation can be obtained from the measured probabilities $|\langle +z| +x\rangle|^2$ and $|\langle +z| +y\rangle|^2$ which yields[1]

$$|+z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle + |-x\rangle), \tag{4a}$$

$$|-z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle - |-x\rangle), \tag{4b}$$

$$|+z\rangle = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle), \tag{4c}$$

$$|-z\rangle = \frac{-i}{\sqrt{2}}(|+y\rangle - |-y\rangle). \tag{4d}$$

For an arbitrary direction $|\pm\vec{n}\rangle$ defined by the angles $\phi$ and $\theta$ we get the transformation [23]

$$|+z\rangle = \cos\left(\frac{\theta}{2}\right)|+\vec{n}\rangle + \sin\left(\frac{\theta}{2}\right)|-\vec{n}\rangle \,, \tag{5a}$$

$$|-z\rangle = e^{-i\phi}\sin\left(\frac{\theta}{2}\right)|+\vec{n}\rangle - e^{-i\phi}\cos\left(\frac{\theta}{2}\right)|-\vec{n}\rangle \,. \tag{5b}$$

Now we can rewrite the entangled state $|\psi^-\rangle$ for example in $x$ direction and get

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|{\downarrow_A}{\uparrow_B}\rangle_z - |{\uparrow_A}{\downarrow_B}\rangle_z) = \frac{1}{\sqrt{2}}(|{\uparrow_A}{\downarrow_B}\rangle_x - |{\downarrow_A}{\uparrow_B}\rangle_x). \tag{6}$$

[1] Note that for ease of notation we choose $|\uparrow\rangle_z \equiv |+z\rangle$, $|\downarrow\rangle_z \equiv |-z\rangle$, etc.

As we can see, the state is still entangled in the new basis. The state we just considered is one of four states of the so called Bell states

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|\downarrow_A\downarrow_B\rangle_z + |\uparrow_A\uparrow_B\rangle_z), \tag{7a}$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|\downarrow_A\downarrow_B\rangle_z - |\uparrow_A\uparrow_B\rangle_z), \tag{7b}$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|\downarrow_A\uparrow_B\rangle_z + |\uparrow_A\downarrow_B\rangle_z), \tag{7c}$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|\downarrow_A\uparrow_B\rangle_z - |\uparrow_A\downarrow_B\rangle_z). \tag{7d}$$

This is analogue to the singlet and triplet states but uses entangled states. Later, in section 1.1.3 we will see that these states are 'maximally entangled' states as they are entangled in all degrees of freedom.

### 1.1.2 The Density Operator - Mixed States

For many systems such as the one in the example above, we would be only interested in one of the degrees of freedom, e.g. polarization. Furthermore, we would like to give an expression for e.g. the unpolarized light.

Let us consider a hermitian operator $\hat{A} = \sum_i a_i |a_i\rangle \langle a_i|$ representing an observable with possible result values $a_i$ and states $|a_i\rangle$. The probability to measure $a_i$ in a pure state $|\psi\rangle$ is given by

$$W(a_i) = |\langle a_i|\psi\rangle| = \langle a_i|\psi\rangle \langle\psi|a_i\rangle . \tag{8}$$

Now we could interpret $|\psi\rangle\langle\psi| \equiv \rho_\psi$ as an operator to express the above probability. In order 'mix' several different states, one can take a linear combination which leads to the

**Definition 1.2** (Density Operator)**.**

$$\hat{\rho} = \sum_\nu p_\nu |\phi_\nu\rangle \langle\phi_\nu| \quad \textit{where} \quad \sum_\nu p_\nu = 1 \tag{9}$$

Analogue to eq. (8) we can get the probability for a measurement outcome $a_i$ from

$$W(a_i) = |\langle a_i|\,\hat{\rho}\,|a_i\rangle| = \sum_\nu p_\nu |\langle a_i|\phi_\nu\rangle|. \tag{10}$$

From this, we can obtain the expectation value $\langle\hat{A}\rangle$ as

$$\langle\hat{A}\rangle = \sum_i a_j\, W(a_i) = \sum_i a_j \operatorname{tr} \hat{\rho}\, |a_i\rangle \langle a_i| = \operatorname{tr}(\hat{\rho}\hat{A}). \tag{11}$$

As we have seen, for any pure state, the density operator can be written as $\hat{\rho} = |\psi\rangle\langle\psi|$. Let us again consider an entangled state with two degrees of freedom (here in spin) such as $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle)$. When particle $A$ is measured as $|\uparrow\rangle$, particle $B$ will be measured $|\downarrow\rangle$ and vice versa. The reduced density matrix is given by

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle)(\langle\uparrow_A\downarrow_B| - \langle\downarrow_A\uparrow_B|) \tag{12}$$

Now, one could ask, what if an operator $\hat{A}$ acts only on system $A$, but not on 'the rest' (or environment) $B$? The most general state can be given by the density matrix

$$\hat{\rho}_{AB} = \sum_{a,a'}\sum_{b,b'} p_{aa'bb'}\,|a,b\rangle\langle a',b'|\,. \tag{13}$$

Now we consider an operator acting only on $A$, such that $\hat{A} = \mathbb{1}_B \otimes \sum_{aa'} A_{aa'}\,|a\rangle\langle a'|$. For the expectation value we get

$$\langle\hat{A}\rangle = \mathrm{tr}_{ab}\hat{\rho}_{ab}\hat{A} = \mathrm{tr}_a\;(\mathrm{tr}_b\hat{\rho}_{ab})\;\hat{A} = \mathrm{tr}_a\;\hat{\tilde{\rho}}_a\;\hat{A} \tag{14}$$

with the

**Definition 1.3** (Reduced Density Operator)**.**

$$\hat{\tilde{\rho}}_a = tr_b\hat{\rho}_{ab} \quad where \quad \tilde{p}_{aa'} = \sum_b p_{aa'bb} \tag{15}$$

If we now apply this to the density operator given in eq. (12) for the pure state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle)$, we get

$$\begin{aligned}\hat{\tilde{\rho}}_A &= \frac{1}{2}\mathrm{tr}_B\;(|\uparrow_A\downarrow_B\rangle\langle\uparrow_A\downarrow_B| - |\downarrow_A\uparrow_B\rangle\langle\uparrow_A\downarrow_B| - |\uparrow_A\downarrow_B\rangle\langle\downarrow_A\uparrow_B| + |\downarrow_A\uparrow_B\rangle\langle\downarrow_A\uparrow_B|)\\ &= \frac{1}{2}(|\uparrow_A\rangle\langle\uparrow_A| + |\downarrow_A\rangle\langle\downarrow_A|)\end{aligned} \tag{16}$$

As we can see, the resulting state is a mixed state with equal probability to measure up or down. If we reduce to one part of the system we do not retain any information about the entangled state. This special property of entangled states will become important later.

### 1.1.3 Entanglement Entropy

In order to classify entangled states we would like to measure how 'strong' they are entangled. Analogue to the classical entropy ($S = k_B \cdot n\ln(n)$), van Neumann introduced the entanglement entropy between two systems $A$ and $B$ based on the reduced density matrix as

**Definition 1.4** (Entanglement Entropy)**.** *The entanglement entropy between two systems A and B with the reduced density matrix $\hat{\tilde{\rho}}_A$ is given by*

$$S_A = -tr_A \left(\hat{\tilde{\rho}}_A \ln(\hat{\tilde{\rho}}_A)\right) \equiv S_B \tag{17}$$

However, if the state is non-diagonal, this expression is difficult to evaluate. In order to do so, we can use the so called Schmidt-Decomposition of the state to get orthonormal states. Using this decomposition we will see that we can obtain a diagonal state from the two original states by applying a basis transformation. [23, 22]

Consider two systems $A$ and $B$ with orthonormal basis states $|a\rangle$ and $|b\rangle$ respectively. Any state in these two systems can be written in the form

$$|\psi\rangle = \sum_{a,b} c_{ab} |a\rangle \otimes |b\rangle \,. \tag{18}$$

Now the coefficients $c_{ab}$ can be regarded as a matrix. Using singular value decomposition (SVD) (see [23] for an explanation) this matrix can be rewritten by using two unitary matrices $U, V$ and a diagonal matrix $\Lambda$ as

$$(c_{ab}) = U\Lambda V^{\dagger}, \tag{19}$$

With the resulting dimensionality of $\chi \leq \min(\dim(A), \dim(B))$. Applying this to eq. (18) yields

$$|\psi\rangle = \sum_{a,b}\sum_{\gamma=1}^{\chi} \lambda_\gamma (V^\dagger)_{\gamma b} |a\rangle \otimes |b\rangle = \sum_{\gamma=1}^{\chi} \lambda_\gamma \underbrace{\sum_a U_{a\gamma} |a\rangle}_{|A_\alpha\rangle} \otimes \underbrace{\sum_b (V^\dagger)_{\gamma b} |b\rangle}_{|B_\alpha\rangle} \,. \tag{20}$$

The last step can be understood as using a unitary basis transformation for $|a\rangle$ and $|b\rangle$ to the basis $|A\rangle$ and $|B\rangle$ where the state is diagonal such that

$$|\psi\rangle = \sum_{\gamma=1}^{\chi} \lambda_\gamma |A_\alpha\rangle \otimes |B_\alpha\rangle \quad \text{where} \quad \chi \leq \min(\dim(A), \dim(B)). \tag{21}$$

It must be noted that the '*Schmidt-Decomposition cannot, in general, be extended to more than two subsystems*' [22]. Furthermore, the decomposition is not unique, such that multiple decompositions can exist for the same state. For a more thorough explanation and more details see e.g. [22, 15].

Using the Schmidt-Decomposition we can obtain the reduced density matrix for subsystem $A$ as

$$
\begin{aligned}
\hat{\hat{\rho}}_A &= \mathrm{tr}_B \hat{\rho} = \mathrm{tr}_B \sum_{\gamma,\delta=1}^{\chi} \lambda_\gamma \lambda_\delta \left| A_\gamma \right\rangle \left| B_\gamma \right\rangle \left\langle A_\delta \right| \left\langle B_\delta \right| \\
&= \sum_{\eta} \left\langle B_\eta \right| \sum_{\gamma,\delta=1}^{\chi} \lambda_\gamma \lambda_\delta \left| A_\gamma \right\rangle \left| B_\gamma \right\rangle \left\langle A_\delta \right| \left\langle B_\delta \right| \left| B_\eta \right\rangle \\
&= \sum_{\gamma=1}^{\chi} \lambda_\gamma^2 \left| A_\gamma \right\rangle \left\langle A_\gamma \right| .
\end{aligned}
\tag{22}
$$

Now we insert this into the entanglement entropy (definition 1.4) and obtain

$$
S_A = -\sum_{\gamma=1}^{\chi} \lambda_\gamma^2 \ln \left( \lambda_\gamma^2 \right) \equiv S_B \tag{23}
$$

Note that the entanglement entropy for System $A$ relative to $B$ is the same as $B$ relative to $A$. Furthermore, the entropy approaches its maximum if $\lambda_\gamma = \lambda_\delta \ \forall \ \ \gamma, \delta \ \in \{1, \ \dots, \chi\}$ such that $\lambda_\gamma^2 = \frac{1}{\chi}$. The resulting entropy is then given by

$$
S_{A,max} = -\chi \frac{1}{\chi} \ln \frac{1}{\chi} = \ln \chi \tag{24}
$$

We have already seen an example for such a so called maximally entangled state, the Bell state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|\downarrow_A \uparrow_B\rangle_z - |\uparrow_A \downarrow_B\rangle_z)$. As we have already seen in eq. (16), the reduced density matrix is already diagonal, such that the entropy $S_A(|\psi^-\rangle) = 2\frac{1}{2} \ln \frac{1}{2} = \ln 2$. Since the dimensionality of the Hilbertspace $\chi = 2$, the state is maximally entangled. [23]

Entangled states were first introduced by EPR to show their paradox, which will be discussed in the following section.

## 1.2 EPR Paradoxon

In 1935 Einstein, Podolsky and Rosen (EPR) discussed in their paper the so called EPR-Paradox [21]. This '*was advanced as an argument that QM could not be a complete theory, but should be supplemented by additional variables*' [7].

The EPR paradox and according assumptions were often misinterpreted, as Bell discussed based on the Einstein-Born letters [6]. In the following, the EPR paradox together with the underlying assumptions shall be briefly presented (for a more precise discussion of the concepts involved see e.g. [10, p119 ff]). For the discussion we will follow [21], [6], [56].

For their paradox, EPR considered two observers (or experiments), which we will call Alice (A) and Bob (B). The setup is shown in the Minkowsky diagram in fig. 1, where the two spacetime regions of Alice and Bob are drawn as spacelike separated (such that they can not exchange information after a certain point in time).

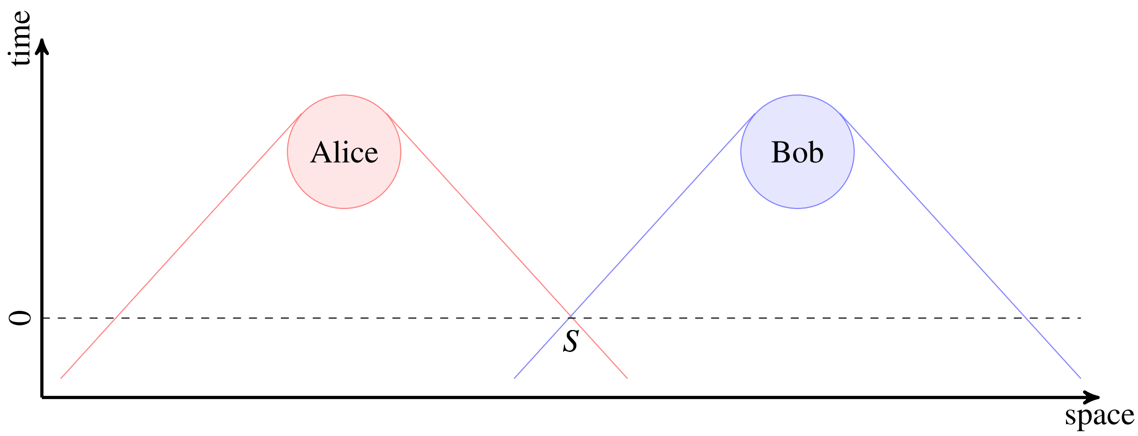



Figure 1: Minkowsky (spacetime) diagram considered by EPR. The diagonal lines indicate the the backwards light cones of the spacetime regions of Alice and Bob, where $S$ is a point in spacetime common to both backwards light cones. [10, p122]

The EPR argument now assumes that we know precisely the state of the whole system until an arbitrary point in time $t = 0$. From this time on, no interaction between Alice and Bob is permitted, since all interactions (or information exchanges) must propagate with the speed of light or slower and are ruled out by the spacelike separation.

Now EPR assume that quantum mechanics is a *complete theory* or in other words that all elements of reality are (fully) represented elements in QM. Furthermore, they call a quantity *real*, when they can predict a quantity without disturbing a system. This assumption comes from the classical understanding that if we drop a stone, we can predict it's velocity without measuring it - we can be sure that the stone is falling without observing it.

The contradiction appears when we consider an entangled state, known at $S$ and shared between

Alice and Bob. When Alice does not measure it's part of the state, QM can not predict what the (resulting) state would be after Alice would measure it. Hence, in the sense of EPR it is not real. But if Bob measures his part of the state, QM can predict the state of Alice's particle. The contradiction here is that Alice and Bob are spacelike separated, meaning that after Bobs measurement the state (or the measured value) at Alice should still be the same in a classical understanding but is *not real* (or uncertain) before Bob's measurement and *real* (or predictable) after Bob's measurement without any interaction.

To illustrate the argument better the original argument based on position and momentum operators shall be presented and then a simplified variant after Bohm [13] shall be given.

**Example 1:** The original argument given by EPR assumes a *complete theory* where '*every element of the physical reality must have a counterpart in the physical theory.*' [21]. *Reality* is stated as: '*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to that physical quantity.*' [21]

Based on those two assumptions, they argue that two quantities described by non commuting operators (e.g. for position and momentum operators $[\hat{Q}, \hat{P}] = i\hbar$) can not have simultaneous physical reality. Because, if one quantity is known precisely, the other can not be measured precisely (Heisenberg's Principle of Uncertainty, e.g. [43, p. 6]).

Consider now two systems $A$ and $B$[2]. Suppose the state of the whole system at time $t = 0$ is known and interaction between the systems is only permitted for $t \in [0, T]$. No interaction for $t > T$ means that causes of all events in the system must be local, e.g. in the same system, following the principle of relativity.

Now suppose we choose a quantity $\hat{P}$ in system $A$ with the eigenstates $|a_i\rangle$. The state of the system can then be written as $\psi = \sum_{i=1}^{I} c_i |a_i\rangle |b_i\rangle$, where $c_i |b_i\rangle$ can be considered coefficients for the expansion in $|a_i\rangle$. Measuring this quantity at $t > T$, we will obtain a state $c_m |a_m\rangle |b_m\rangle$.

If instead we chose a different quantity $\hat{Q}$ with eigenstates $|\alpha_k\rangle$ we obtain the expansion $\psi = \sum_{k=1}^{K} \gamma_k |\alpha_k\rangle |\beta_k\rangle$. Measuring $\hat{Q}$ at $t > T$, we will obtain a state $\gamma_n |\alpha_n\rangle |\beta_n\rangle$.

As a consequence, depending on the quantity we chose to measure in $A$, with the results $|a_m\rangle$ and $|\alpha_n\rangle$, system $B$ is left in two different states. However, the two systems do not interact. Therefore, EPR argue, no real change in the second system can take place, meaning it is possible to assign two different wave functions to the second system if $|a_m\rangle$ is independent of $|\alpha_n\rangle$. Now, $|a_m\rangle$ and $|\alpha_n\rangle$ are eigenfunctions of two operators, $\hat{P}$ and $\hat{Q}$ respectively, where $[\hat{Q}, \hat{P}] = i\hbar$. As we have assumed before, without disturbing the other system, either $\hat{P}$ or $\hat{Q}$ can be measured. According to the assumption of *reality*, when the quantity corresponding to $\hat{P}$ is measured at $A$, $|\alpha_n\rangle$ must be considered an element of reality. However, $|\alpha_n\rangle$ and $|a_m\rangle$ must belong to the same reality, as we could choose to measure $\hat{Q}$ at $B$ and obtain $|a_m\rangle$. EPR conclude '*Thus it is possible to assign two different wave functions to the same reality.*

[2] The original argument is given in position space. Here, Dirac notation shall be used.

*... we proved that either (1) the QM description of reality can not be complete or (2) when two operators corresponding to two physical quantities do not commute, the two quantities cannot have simultaneous reality.*' [21]. If it is assumed that the wave function gives a complete description of reality, two quantities corresponding to two operators can have simultaneous reality. However, the last statement is in conflict with $[\hat{Q}, \hat{P}] \neq 0$. [21]

This argument however, is rather prone to misunderstanding, as Bell [6] has shown with the example of the Einstein-Born correspondence. Therefore this subtle point must be made clear: The EPR argument is not for 'determinism' but for the *principle of local causality*, meaning that all effects must have *local* causes limited by the speed of light. As Bell clearly states, '*Einstein had no difficulty accepting that affairs in different places could be correlated. What he could not accept was that an intervention at one place could influence, immediatly, affairs at the other.*' [6]

**Example 2:** Suppose the state of the whole system at time $t = 0$ is known as $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle)$. Furthermore, interaction between $A$ and $B$ is only permitted for $t \in [0, T]$. Then, when $A$ is measured, for any $t > T$, the state of of $B$ depends on the measured value of $A$ (e.g. when $|\uparrow_A\rangle$ is measured, $B$ will be in the state $|\downarrow_B\rangle$). Therefore, after the measurement in $A$ we can predict the result at $B$ (see[13, p611 ff])

But, one must ask - how does this differ from probability theory? Bell gives the example of Bertlmann's socks.

> *Dr. Bertlmann likes to wear two socks of different colors. Which color he will have on a given foot on a given day is quite unpredictable. But when you see (fig. 2) that the first sock is pink, you can already be sure that the second one will not be pink. Observation of the first and experience of Bertlmann gives immediate information about the second. There is no accounting for tastes, but apart from that, there is no mystery there. And is not the EPR business just the same? [6]*

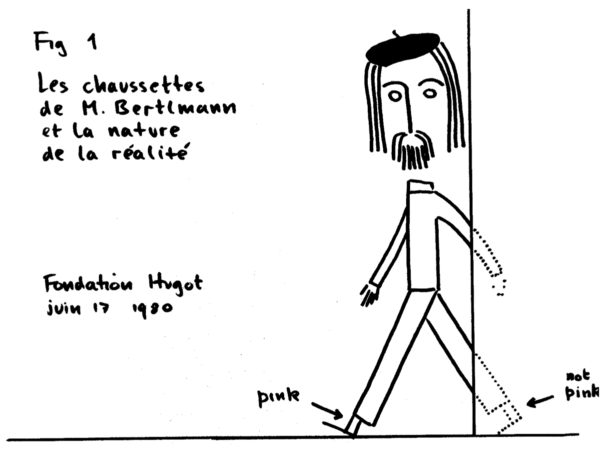



Figure 2: Cartoon by Bell in the CERN preprint Ref.TH.2926-CERN [6] from 1980

If one considers the second example given, the answer must necessarily be yes. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be the spin states in $z$ direction (both $A$ and $B$ measure in the same basis). Then, we can show that for either $A$ or $B$ the reduced state is $|\hat{\rho}_{red}\rangle = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$. This means, the probability for either option is $P(|\uparrow\rangle) = P(|\downarrow\rangle) = \frac{1}{2}$, which is equivalent to two independent coin flips. Only if information between the two systems is exchanged (which must happen, due to locality, with subluminal speed), the correlation appears in the form $|\uparrow_A\rangle \Leftrightarrow |\downarrow_B\rangle$ and $|\downarrow_A\rangle \Leftrightarrow |\uparrow_B\rangle$. However, this is Bell's argument, because this does not differ from the idea given in fig. 2. Here, the correlation would be $|pink_{leftfoot}\rangle \Leftrightarrow |not\,pink_{rightfoot}\rangle$ and vice versa, which is perfectly compatible with any classical theory. The difference to the EPR paradox lies within the assumption of just one basis for both measurements. In the argument given by EPR, the two systems are measured in two different basis. To continue with the example of Spin, $A$ and $B$ could each choose any two directions $\vec{a}$ and $\vec{b}$ to measure the spin randomly. As we will see in section 2.2, even with the assumption of any number of local hidden variables we can not describe this correlation under the given assumptions.

## 1.3 A contradiction between QM and the EPR-Argument using GHZ states

Bells original derivation used probabilities to derive an inequality. More accessible is a contradiction presented later by Greenberger, Horne, and Zeilinger [29] based on multi-particle entangled states, called the Greenberger, Horne and Zeilinger (GHZ)-states. We will see that the EPR assumptions stated earlier will contradict the quantum mechanical result. [9, p195 f]

We consider a three particle spin state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\uparrow_B\uparrow_C\rangle_z - |\downarrow_A\downarrow_B\downarrow_C\rangle_z) \tag{25}$$

given in the $z$-basis, as indicated in the index. A basis transformation in x can be done using the relations given in eq. (4) with the result

$$|\psi\rangle = \frac{1}{2}(|\uparrow_A\uparrow_B\downarrow_C\rangle_x + |\uparrow_A\downarrow_B\uparrow_C\rangle_x + |\downarrow_A\uparrow_B\uparrow_C\rangle_x + |\downarrow_A\downarrow_B\downarrow_C\rangle_x). \tag{26}$$

Now associate the measurement value $\{\sigma_x\}_A$ of the particle $A$ with

$$\{\sigma_x\}_A = \begin{cases} +1 & \text{for } |\uparrow_A\rangle \\ -1 & \text{for } |\downarrow_A\rangle \end{cases} \tag{27}$$

and similar for $B$ and $C$. Using this, the product of the outcomes $\{\sigma_x\}_A\{\sigma_x\}_B\{\sigma_x\}_C$ will always be $-1$ for the state (26). Now, if we measure $B$ and $C$ in $y$ direction and $A$ in $x$ we need to express $|\psi\rangle$ which reads

$$|\psi\rangle = \frac{1}{2}(|\uparrow_A\rangle_x |\uparrow_B\uparrow_C\rangle_y + |\uparrow_A\rangle_x |\downarrow_B\downarrow_C\rangle_y + |\downarrow_A\rangle_x |\downarrow_B\uparrow_C\rangle_y + |\downarrow_A\rangle_x |\uparrow_B\downarrow_C\rangle_y\,. \tag{28}$$

Therefore any product of outcomes yield $\{\sigma_x\}_A\{\sigma_y\}_B\{\sigma_y\}_C = 1$. The same is also true for any permutation, as we could just exchange the indices for the different particles. In total this gives us

$$\{\sigma_x\}_A^I\{\sigma_x\}_B^I\{\sigma_x\}_C^I = -1, \tag{29a}$$

$$\{\sigma_x\}_A^{II}\{\sigma_y\}_B^{II}\{\sigma_y\}_C^{II} = 1, \tag{29b}$$

$$\{\sigma_y\}_A^{III}\{\sigma_x\}_B^{III}\{\sigma_y\}_C^{III} = 1, \tag{29c}$$

$$\{\sigma_y\}_A^{IV}\{\sigma_y\}_B^{IV}\{\sigma_x\}_C^{IV} = 1. \tag{29d}$$

Note the indices $I - IV$, as the different measurement results are not the same in quantum mechanics since they depend on the other two systems (e.g. $\{\sigma_x\}_A^I$ depends on $\{\sigma_x\}_B^I\{\sigma_x\}_C^I$, since if we know the result from $B$ and $C$ we also know the result of $A$.

But now according to EPR we could choose to measure the particles in non interacting (spacelike separated) systems. Now the measurement outcome in the system $A$ may not depend on anything in the other two systems, $\{\sigma_x\}_A^I = \{\sigma_x\}_A^{II}$, etc. If we now multiply the above equations 29, we get the obvious contradiction

$$(\{\sigma_x\}_A\{\sigma_x\}_B\{\sigma_x\}_C\{\sigma_y\}_A\{\sigma_y\}_B\{\sigma_y\}_C)^2 = -1. \tag{30}$$

This can not be true, as a square of real numbers can not be $-1$. Therefore it is obvious that the condition $\{\sigma_x\}_A^I = \{\sigma_x\}_A^{II}$ derived from the assumptions of the EPR argument must be wrong.

The first experiments based on multi-particle entanglement have been performed in 1999, see e.g. [42], where these correlations (eq. (29)) could also be observed. However, experiments violating inequalities are easier to achieve and have been achieved far earlier, as the preparation of three particle entangled states is more difficult. But opposing to the statistical correlation shown by an inequality, multi-particle states are able to show the correlation with every single experiment.

In the following chapter, we will see how to derive such inequalities, as well as take a deeper look into the assumptions they are based on.

# 2 Bells Theorem

## 2.1 Overview

As we have seen in section 1, contradictions between QM and the assumptions by EPR can have different forms and are usually collected under the name 'Bells Theorem'. Different versions based on better founded as well as relaxed assumptions have been given over the years [41]. With improvements of these assumptions as well as additional assumptions, experimental tests were be performed, showing that the assumptions made by EPR (in the frame of the other assumptions) also contradict with nature. With the improvement of theory better and more stringent experiments could be performed.

Additionally, also a philosophical aspect is attached to Bell's Theorem. Leading up from the philosophical discussions of EPR, there is still a lot of ongoing discussion, mainly focused on the interpretation and to get a clearer (mathematical) picture of 'philosophical' concepts such as locality. The matter of interpretation is very related to this issue - dependent on the assumptions used to derive a Bell Inequality, different options are given once its violation has been demonstrated. Some of those assumptions, such as free will are not debatable in the form of a physical theory, as they can not be shown to have any experimental impact. However, of locality and realism (two basic assumptions in almost all of the classical theory) it still remains unclear which to reject (see e.g. Bertlmann and Zeilinger [10, p119 ff]).

In the following sections, at first we will discuss Bell's original idea based on probabilities and systems with only two degrees of freedom. Then we will come closer to the experiment with the improvements of Clauser, Horne, Shimony and Holt (CHSH) and Clauser and Horne (CH). Finally we will arrive at the CH-Eberhard inequality, which allows for *detection loophole* free experiments.

This chapter is mainly based on Bell [7], Bertlmann and Zeilinger [10, p122 ff], Clauser et al. [19], Clauser and Horne [18] and Eberhard [20].

## 2.2 Bell Inequality (1964)

Before considering a more generalized case of Bells Theorem, a specialized variant based on Bertlmann [11], Bell [7], Bertlmann and Zeilinger [10, p122 ff] will be presented here. Let us consider again a source of singlet state particles, which propagate to two (space-like) separated measurement stations Alice ($A$) and Bob ($B$) such as shown in fig. 3.

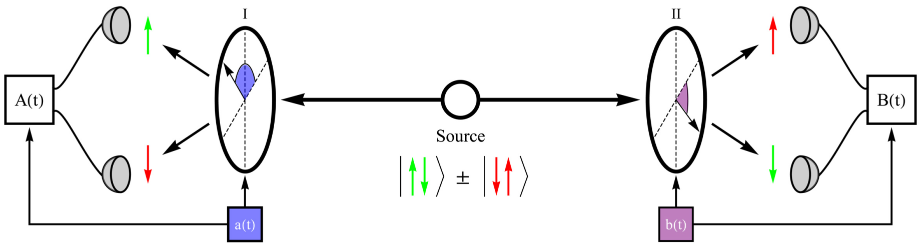



Figure 3: Bohm-EPR setup of a pair of spin $\frac{1}{2}$ particles in the singlet state. The particle propagates to the two (space-like) separated stations called Alice ($A$) and Bob ($B$), where its spin is measured in the directions $\vec{a}$ and $\vec{b}$ respectively. Graphic from Bertlmann [11].

A measurement may for example be performed by a Stern-Gerlach apparatus oriented along a direction $\vec{a}$ on the components $\hat{\vec{S}}_A$ and $\hat{\vec{S}}_B$. According to QM, if the measurement of $\hat{\vec{S}}_A \cdot \vec{a}$ yields +1, then the measurement of $\hat{\vec{S}}_B \cdot \vec{a}$ must yield $-1$ and vice versa. The EPR argument, according to Bell [7] goes the following:

> *Since we can predict in advance the result of measuring any chosen component of $\hat{\vec{S}}_B$, by previously measuring the same component of $\hat{\vec{S}}_A$, it follows that the result of any such measurement must actually be predetermined. Since the initial quantum mechanical wave function does not determine the result of an individual measurement, this predetermination implies the possibility of a more complete description of the state [7].*

This complete description shall be denoted by an additional parameter $\lambda$. The parameter $\lambda$ may describe any set of parameters (continuous or not) or functions, and may even depend on the the measurement settings $\vec{a}$,$\vec{b}$ [7].

Based on the EPR argument, two assumptions[3] are made. Here we will use the definitions from Bertlmann and Zeilinger [10, 122 ff]. The first condition is *predetermination*, where we assume that the hidden variable $\lambda$ determines the outcome of a measurement.

[3] One should be very careful about the labelling of the assumptions and definitions of these labels in different sources. For a discussion see e.g. [10, p119ff] or [41] for a different point of view.

**Definition 2.1** (PREDETERMINATION). *The result of an individual measurement is predetermined by the parameter $\lambda$.*[a]

[a] This is also sometimes labelled *realism* or *outcome determinism*, see e.g. [10, p278] or [41] respectively.

If we measure the observable $\hat{\vec{S}} \cdot \vec{a}$ the expectation value is given by the relative frequency of observation times the value measured. For the continuous variable $\lambda$ the relative frequency is given by the probability distribution $\rho(\lambda)$ where we assume $\int d\lambda \rho(\lambda) = 1$. The expectation value is then given by

$$P(A \mid \vec{a}, \vec{b}, S) = \int d\lambda \rho(\lambda) A(\vec{a}, \vec{b}, S, \lambda) \tag{31}$$

where the measurement outcome in dependence on the settings, source $S$ and $\lambda$ is denoted by $A(\vec{a}, \vec{b}, S, \lambda)$

Now we assume *locality* (formulation from Bertlmann and Zeilinger [10, p119ff]), here formulated for Alice's result $A$.

**Definition 2.2** (LOCALITY). *The result of a measurement in one system is unaffected by operations on a distant system.*[abc]

$$\forall \vec{b}, A, \vec{a}, \lambda \quad P(A \mid \vec{a}, \vec{b}, S) = P(A \mid \vec{a}, S)$$

[a] The term locality is sometimes defined differently or confused with what [10, p119ff] calls *signal locality*, *local causality* or *bell locality*.

[b] The definition used here is sometimes also *parameter independence*, see e.g. [41] for a discussion.

[c] Here formulated as Alice's outcome is independent of Bobs settings $\vec{b}$.

Note that this also implies $A(\vec{a}, \vec{b}, S, \lambda) = A(\vec{a}, S, \lambda)$ which will be denoted as $A(\vec{a}, \lambda)$.

Furthermore we assume the measurements Alice and Bob perform yield values

$$A(\vec{a}, \lambda) = \pm 1 \text{ and } B(\vec{b}, \lambda) = \pm 1. \tag{32}$$

Equivalent to the expectation value above, Bell [7] gives the expectation value of the product of the two components as

$$P(\vec{a}, \vec{b}) \equiv P(A, B \mid \vec{a}, \vec{b}, S) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) \tag{33}$$

where $\rho(\lambda)$ again denotes the probability distribution of $\lambda$, which is assumed to be normalized such that $\int d\lambda \rho(\lambda) = 1$.

Together above assumptions yield Bells Theorem or Bell's inequality:

**Theorem 2.3** (Bell-1964)**.** *A model satisfying* Predetermination *and* Locality *must comply with the inequality*

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \leq 1 + P(\vec{b},\vec{c}).$$

*There exist QM phenomena, for which there is no model satisfying these assumptions. [10, 122 ff][7]*

*Proof.* [7] Based on eq. (32), $P(\vec{a},\vec{b}) \geq -1$ in eq. (33) and $P(\vec{a},\vec{b}) = -1$ only for $\vec{a} = \vec{b}$ and

$$A(\vec{a},\lambda) = -B(\vec{a},\lambda) \tag{34}$$

except at points $\lambda$ with zero probability. Note that eq. (34) implies perfect anticorrelation of the measurement results for the same measurement settings $\vec{a}$ at $A$ and $B$. This yields

$$P(\vec{a},\vec{b}) = -\int d\lambda\rho(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda) \tag{35}$$

When another unit vector $\vec{c}$ is introduced, from $A(\vec{b},\lambda)A(\vec{b},\lambda) = 1$ follows that

$$\begin{aligned} P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) &= -\int d\lambda\rho(\lambda)\left[A(\vec{a},\lambda)A(\vec{b},\lambda) - A(\vec{a},\lambda)A(\vec{c},\lambda)\right] \\ &= \int d\lambda\rho(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda)\cdot[A(\vec{a},\lambda)A(\vec{c},\lambda) - 1] \end{aligned} \tag{36}$$

Furthermore, since $P(\vec{a},\vec{b}) \geq -1$

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \leq \int d\lambda\rho(\lambda)\,[A(\vec{a},\lambda)A(\vec{c},\lambda) - 1] = 1 + P(\vec{b},\vec{c}) \tag{37}$$

With this we have obtained the inequality stated in theorem 2.3.

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \leq 1 + P(\vec{b},\vec{c}) \tag{38}$$

Now we have to show a QM state violating this inequality, therefore showing that QM is incompatible with the combination of assumptions leading to this inequality. [10, 43 ff]. Based on the singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle)$, the quantum mechanical expectation value is defined as

$$P(\vec{a},\vec{b}) = \langle\psi^-|\,\hat{\vec{S}}_A\cdot\vec{a}\otimes\hat{\vec{S}}_B\cdot\vec{b}\,|\psi^-\rangle = -\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|cos(\theta), \tag{39}$$

where $\theta$ is the angle between the two vectors $\vec{a}$ and $\vec{b}$. Now we choose one set of measurement parameters or directions in case of spin. These can be chosen such that e.g. $\vec{a} \cdot \vec{c} = 0$ and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \frac{1}{\sqrt{2}}$. This yields the contradiction

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| - P(\vec{b}, \vec{c}) = \sqrt{2} \leq 1 \ \lightning \tag{40}$$

'*Then for at least one quantum mechanical state ... the statisitcal predictions of quantum mechanics are incompatible with separable determinism.*'[7] □

As Bell [7] states, we have proven that

> *In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measurement device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant.*

Or in other words, the QM predictions hold and we assume a model with *predetermination*, we must necessarily sacrifice the assumption of locality or vice versa [10, 122 ff].

## 2.3 CHSH Inequality

A generalization of Bell's inequality was proposed in 1969 by Clauser, Horne, Shimony and Holt (CHSH), because the original inequality relied on perfect anti-correlation (eq. (34)) of the observed measurements which is impossible to reproduce in any (imperfect) experiment. Here we will follow the reasoning of [19] to derive an inequality which does not assume perfect anti-correlation. Furthermore we will see how we can adapt this inequality to allow for detection inefficiency of the apparatus.

The considered experimental situation is again shown in fig. 4

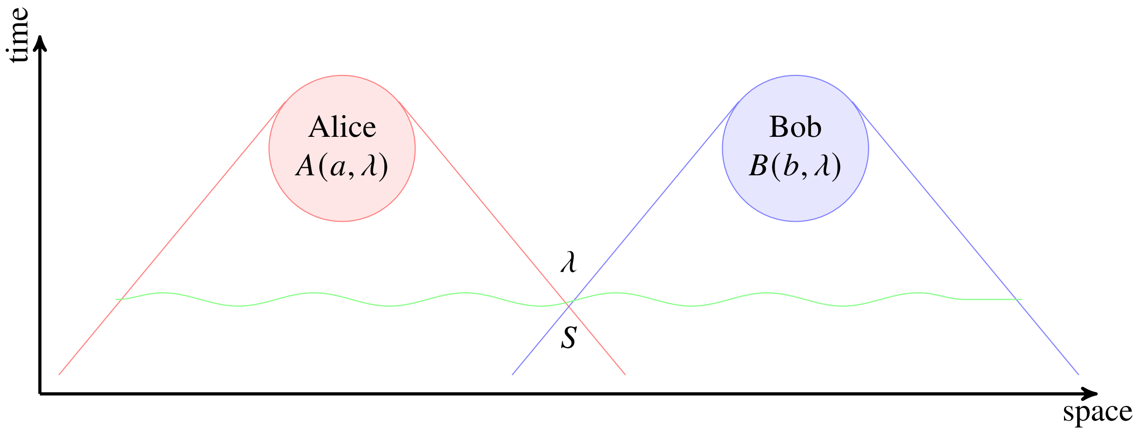



Figure 4: Minkowsky (spacetime) diagram of the considered experimental setup. The diagonal lines indicate the the backwards light cones of the spacetime regions of the two experimenters, Alice and Bob, where $A(a, \lambda)$ denotes the measurement result $A$ dependent on the free parameter $a$ and the hidden variable $\lambda$. Analogue for $B$, where $A$ and $B$ share the common source (or cause) $S$. [10, p122]

We again consider two particles moving to the Experimenters Alice and Bob, where they are measured with the results $A$ and $B$, depending on the (free) parameters $a$ and $b$ respectively. We will label binary results of each those performed measurements $+1$ and $-1$.

Furthermore we assume that the results $A$ and $B$ also depend on the hidden parameters $\lambda$, where locality requires the that $B$ is independent of $a$ and vice versa, since Alice and Bob could be spacelike separated, as shown in fig. 4. From these assumptions using the measurement settings $a, b, b', c$, the CHSH inequality can be derived.

**Theorem 2.4** (CHSH-1969). *A model satisfying* PREDETERMINATION *and* LOCALITY *must comply with the inequality*

$$|P(a,b) - P(a,c)| \leq 2 - P(b',c) - P(b',c)$$

*There exist QM phenomena, for which there is no model satisfying these assumptions [19].*

*Proof.* Analogue to Bell's derivation CHSH again consider the correlation function

$$P(a,b) = \int_\Gamma A(a,\lambda)B(b,\lambda)\rho(\lambda)d\lambda \tag{41}$$

where $\Gamma$ is the total $\lambda$ space with the normalization $\int_\Gamma \rho(\lambda)d\lambda = 1$. Furthermore we can again introduce another setting $c$ and arrive with the use of $P(a,b) \geq -1$ at the result (similar to eq. (37))

$$\begin{aligned} |P(a,b) - P(a,c)| &\leq \int_\Gamma |A(a,\lambda)B(b,\lambda) - A(a,\lambda)B(c,\lambda)|\rho(\lambda)d\lambda \\ = \int_\Gamma |A(a,\lambda)B(b,\lambda)|[1 - B(b,\lambda)B(c,\lambda)]\rho(\lambda)d\lambda &= 1 - \int_\Gamma B(b,\lambda)B(c,\lambda)\rho(\lambda)d\lambda. \end{aligned} \tag{42}$$

In Bell's derivation we used that for two different settings $b, b'$ the correlation function can reach $P(b',b) = 1$ (see eq. (34), where $P(a,a) = 1$), e.g. perfect anti-correlation. Experimentally more interesting are cases where $P(b',b) = 1 - \delta$, where $\delta \in [0,1]$. Now we can divide $\Gamma$ into two regions $\Gamma_+, \Gamma_-$ such that $\Gamma_\pm = \{\lambda | A(b',\lambda) = \pm B(b,\lambda)\}$. Hence,

$$\begin{aligned} P(b',b) = \int_\Gamma A(b',\lambda)B(b,\lambda)\rho(\lambda)d\lambda &= \int_{\Gamma_+} \rho(\lambda)d\lambda - \int_{\Gamma_-} \rho(\lambda)d\lambda \\ = \int_\Gamma \rho(\lambda)d\lambda - 2\int_{\Gamma_-} \rho(\lambda)d\lambda &= 1 - 2\int_{\Gamma_-} \rho(\lambda)d\lambda = 1 - \delta \\ &\Rightarrow \int_{\Gamma_-} \rho(\lambda)d\lambda = \frac{1}{2}\delta. \end{aligned} \tag{43}$$

From this follows

$$\int_{\Gamma} B(b,\lambda)B(c,\lambda)\rho(\lambda)d\lambda = \int_{\Gamma} A(b',\lambda)B(c,\lambda)\rho(\lambda)d\lambda - 2\int_{\Gamma_-} A(b',\lambda)B(c,\lambda)\rho(\lambda)d\lambda$$
$$\geq P(b',c) - 2\int_{\Gamma_-} |A(b',\lambda)B(c,\lambda)|\rho(\lambda)d\lambda = P(b',c) - \delta = P(b',c) + P(b',c) - 1$$
$$\Rightarrow 1 - \int_{\Gamma} B(b,\lambda)B(c,\lambda)\rho(\lambda)d\lambda \leq 2 - P(b',c) - P(b',c) \tag{44}$$

Combining eq. (42) and eq. (44) yields

$$|P(a,b) - P(a,c)| \leq 2 - P(b',c) - P(b',c) \tag{45}$$

□

Before we can continue to show the contradiction between QM and the assumptions used to derive the inequality, we will discuss the setup proposed by CHSH, a modified version of an earlier experiment by Kocher and Commins [37] using polarized photons instead of spin $\frac{1}{2}$ particles (electrons). The experimental setup is shown in fig. 5.

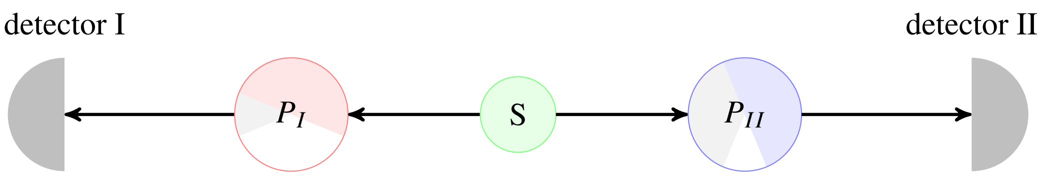



Figure 5: Proposed experimental setup from [19]. Using a (point)-source ($S$) (e.g. the $J = 0 \rightarrow J = 1 \rightarrow J = 0$ cascade of calcium) the entangled photons are analyzed with two polarizers $P_I$ and $P_{II}$ set at two different angles and then detected by the detectors at I and II

Analogue to the orientation of the Stern-Gerlach apparatus, a setting is characterized by the relative angle of the two polarizers. The QM expectation value for this case is given by [23, 41]

$$E(\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta). \tag{46}$$

*Proof.* (continued) Following Clauser et al. [19], we can now show the contradiction of the assumptions with the quantum mechanical expectation value. For the experiment proposed $P(a,b)$ only depends on the difference $b-a$. Hence they replace with the angles

$\alpha = b - a, \beta = c - b$ and $\gamma = b - b'$. This yields

$$|P(\alpha) - P(\alpha + \beta)| \leq 2 - P(\gamma) - P(\beta + \gamma). \tag{47}$$

We associate $P(\theta) \equiv E(\theta)$. This results in

$$|cos(2\alpha) - cos(2(\alpha + \beta))| \leq 2 - cos(2\gamma) - cos(2(\beta + \gamma)). \tag{48}$$

For specific angles, we can maximize the violation of the inequality, specifically with $\alpha = 22.5°, \beta = 45°, \gamma = 157.5°$. This results in the contradiction

$$|\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}| + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\sqrt{2} \leq 2 \ ↯. \tag{49}$$

Hence we can conclude that QM phenomena exist which do not satisfy the assumptions used to derive the inequality. □

The important problem now solved by CHSH is the detection efficiency. Not only was the detection efficiency of any detector low at the time, of all emitted photons from the point source only a cone of half-angle $\theta$ could be collected. Furthermore, the polarizers were also not 100% efficient, as a result, part of the light perpendicular to the polarizer axis will shine through. Hence the count rate measured at each detectors depends on the efficiency of the polarizer (transmission efficiency$\epsilon_M$ in parallel and $\epsilon_m$ perpendicular to the axis of the polarizer) as well as a function of $F(\theta)$. In order to get an inequality suited for the experiment the correlation functions $P(a, b)$ must be replaced by rate counts. This has one major disadvantage, namely that the detection rates have to be divided by a normalization factor (the total rate of emitted pairs of photons in this case), which must be estimated - the so called detection loophole (see 3.3.4). For the derivation of the inequality with count rates refer to [19]. When the optimal angles $\alpha = 22.5^o$, $\beta = 45^o$ and $\gamma = 157.5^o$ for the violation of the inequality 2.4 for the 0-1-0 transition are used (as well as a polarizer transmission efficiency for light perpendicular to the polarizer of $\epsilon_m \cong 10^{-5}$ for calcite detectors), one obtains the condition for the violation of the inequality as

$$\sqrt{2}F(\theta) + 1 < \frac{2}{\epsilon_M} \tag{50}$$

with $F(\theta)$ characterizing the dependency on the collection half-angle $\theta$. Figure 6 illustrates where the inequality 2.4 is violated.

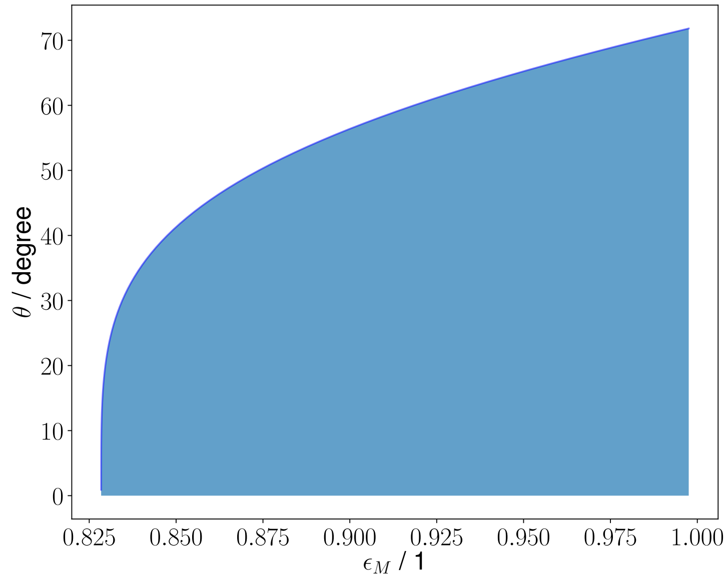



Figure 6: Detector half-angle $\theta$ as a function of the minimum analyzer efficiency $\epsilon_M$. Experiments performed outside of the shaded area can not show a violation of the CHSH inequality. See [19] for an equivalent figure.

The shaded area shows that any experiment performed below a detection efficiency of $\epsilon_M \cong 82.8\%$ can not show a violation of the above inequality [19]. However, this condition could be relaxed under certain circumstances by Eberhard [20] (see below).

CHSH adapted Bell's original inequality, such that the first experiments could be performed by Freedman and Clauser [24]. However, since the total count rate has to be estimated, this experiment is still prone to the so called detection loophole. Essentially, if the detector is particularly selective, one can construct a model where the QM-predictions are satisfied in the frame of a theory satisfying the assumptions made by EPR (see 3.3.4 for a further discussion) [41]. One of the options to deal with this loophole is to derive an equality without the need for a total count rate, as was done by CH in 1974 [18].

## 2.4 CH Inequality

The Clauser and Horne (CH)-inequality was derived as a way to close the detection loophole opened by the (experimental) need for a normalization of the count rates (see also 3.3.4)[41].

This chapter follows Clauser and Horne (CH) [18], who used a slightly different condition for *locality* which Myrvold, Genovese, and Shimony [41] calls *factorizability* (later also used and advocated by Bell [8]).

**Definition 2.5** (FACTORIZABILITY). *The measurement outcome in each system is independent of the other system.*[a]. [b] [c]

$$p_{12}(\lambda, a, b) = p_1(\lambda, a)p_2(\lambda, b)$$

[a] Sometimes also labelled as *Bell locality*, see e.g. [41] or (confusingly) just *locality*, see e.g. [56]

[b] For the relation of this assumption to the previous assumptions / inequalities refer to [18], for two different view points of the different definitions refer to [41] and [10, p119 ff]

[c] Sometimes also labelled as conjunction of parameter independence and outcome independence, see e.g. [41]

Essentially, this is again a condition of locality, as neither random variable depends on the other system. From this condition, in conjunction with the *predetermination* we can again derive a inequality, stated in theorem 2.6.

**Theorem 2.6** (CH-1974). *A model satisfying PREDETERMINATION and FACTORIZABILITY must comply with the inequality*

$$-1 \leq p_{12}(a, b) - p_{12}(a, b') + p_{12}(a', b') - p_1(a') - p_2(b) \leq 0.$$

*There exist QM phenomena, for which there is no model satisfying these assumptions*[a] *[18].*

[a] Without proof. For a setup and measurement settings to show a violation of the inequality refer to [18]

For this proof we will need the following lemma:

**Lemma 2.7.** *Given 6 numbers $x_1, x_2, y_1, y_2, X, Y$ such that*

$$0 \leq x_1 \leq X, \quad 0 \leq x_2 \leq X, \quad 0 \leq y_1 \leq Y \text{ and } 0 \leq y_2 \leq Y.$$

*the function $U = x_1y_1 - x_1y_2 + x_2y_1 + x_2y_2 - Yx_2 - Xy_1$ can be constrained by*

$$-XY \leq U \leq 0$$

*(without proof, see [18], appendix A for the proof).*

*Proof.* For two orientations $a, a'$ of analyzer 1 and $b, b'$ of analyzer 2 the inequalities

$$\begin{aligned} 0 \leq & p_1(\lambda, a) \leq 1, \\ 0 \leq & p_1(\lambda, a') \leq 1, \\ 0 \leq & p_2(\lambda, b) \leq 1, \\ 0 \leq & p_2(\lambda, b') \leq 1, \end{aligned} \tag{51}$$

hold for the usual definition of probabilities $p \in [0, 1]$. Now, using lemma 2.7 we get

$$\begin{aligned} -1 \leq & p_1(\lambda, a)p_2(\lambda, b) - p_1(\lambda, a)p_2(\lambda, b') + p_1(\lambda, a')p_2(\lambda, b) \\ & + p_1(\lambda, a')p_2(\lambda, b') - p_1(\lambda, a') - p_2(\lambda, b) \leq 0. \end{aligned} \tag{52}$$

Furthermore, we apply the *factorizability* condition and obtain

$$\begin{aligned} -1 \leq & p_{12}(\lambda, a, b) - p_{12}(\lambda, a, b') + p_{12}(\lambda, a', b) \\ & + p_{12}(\lambda, a', b') - p_1(\lambda, a') - p_2(\lambda, b) \leq 0. \end{aligned} \tag{53}$$

Finally, multiplying by $\rho(\lambda)$ and integrating over $\lambda$ we gain (analog to eq. (41)) the CH-inequality

$$-1 \leq p_{12}(a, b) - p_{12}(a, b') + p_{12}(a', b) + p_{12}(a', b') - p_1(a') - p_2(b) \leq 0. \tag{54}$$

□

The upper bound of this inequality can now be tested without the need for a normalization factor $N$, if we rewrite in count rates this gives

$$-N \leq N_{12}(a, b) - N_{12}(a, b') + N_{12}(a', b) + N_{12}(a', b') - N_1(a') - N_2(b) \leq 0, \tag{55}$$

where $p_i(a, b) = N_i(a, b)/N$ etc.

As the detection efficiencies at the time were still low, CH introduced another assumption, the so called *no enhancement assumption*, which shall briefly be mentioned. They assume that the detection rate with the analyzer $p_i(a)$ is smaller than the rate of particles detected without the analyzer $p_i(\infty)$. This leads to what is called the second CH-inequality [41]. It can be shown that

$$-1 \leq p_{12}(a, b) - p_{12}(a, b') - p_{12}(a', b) + p_{12}(a', b') - p_1(a') - p_2(b) \leq p_1(\infty) + p_2(\infty). \tag{56}$$

However, as always, this assumption gives rise to another loophole, as models can be constructed satisfying the condition of *factorizability* as well as reproducing the QM predictions. For a proof of the inequality as well as it's violation by quantum mechanics refer to Clauser and Horne [18], for a discussion of the implications refer to Myrvold, Genovese, and Shimony [41].

## 2.5 CH-Eberhard Inequality

Another major improvement to the inequality was made by Eberhard [20] in 1993. Previously we have seen that the minimum detection efficiency for any experiment must be ≥ 82.8%. Eberhard could show that efficiencies ≥ 66.7% suffice, if the background is low enough. This step is important as it enabled the first detection loophole free experiments with photons in 2013 (e.g. the experiment by Giustina et al. [25]) and subsequently with further improvements by Bierhorst [12] to close the memory loophole to the first loophole free experiments (refer to section 3.5). This chapter follows the reasoning presented by Eberhard [20].

The setup considered by Eberhard is equivalent to the setup considered by CHSH. However, instead of rotated polarizers he proposes the use of Nicol prisms, such that the vertical polarized photon ends up in the extraordinary beam and the horizontal photon in the ordinary beam. This is depicted in fig. 7.

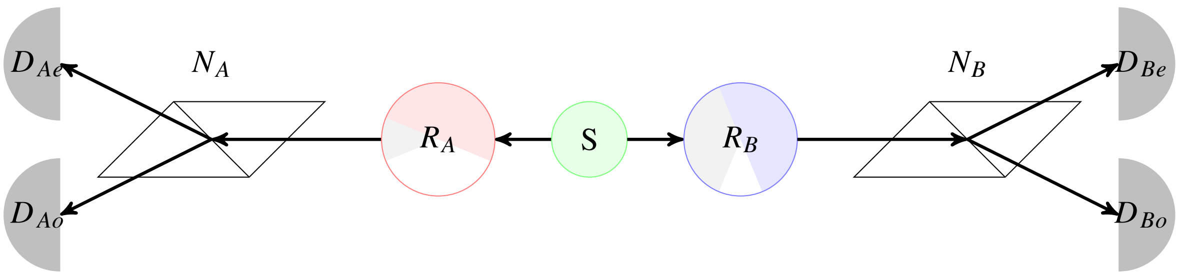



Figure 7: Proposed experimental setup from Eberhard [20]. Using a source ($S$) of entangled photons with two Nicol Prisms $N_A$ and $N_B$ as well as two devices to rotate the incoming photons ($R_A$ and $R_B$, e.g. Plockels cells) and detectors on each side for both the ordinary $D_{Xo}$ and the extraordinary $D_{Xe}$ beam.

The devices $R_X$ can rotate the incoming photons at any angle. For the Bell experiment, four angles, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_1$ are chosen. With this setup, one arrives at the CH-Eberhard inequality.

**Theorem 2.8** (CH-Eberhard-1993)**.** *A model satisfying* P*REDETERMINATION and* L*OCALITY must comply with the inequality*

$$\mathcal{J}_{\mathcal{B}}{}^{ideal} = n_{oe}^{ideal}(\alpha_1, \beta 2) + n_{ou}^{ideal}(\alpha_1, \beta_2) + n_{eo}^{ideal}(\alpha_2, \beta_1) + n_{uo}^{ideal}(\alpha_2, \beta_1) + n_{oo}^{ideal}(\alpha_2, \beta_2) - n_{oo}^{ideal}(\alpha_1, \beta_1) \geq 0.$$

*There exist QM phenomena, for which there is no model satisfying these assumptions*[a] *[20].*

[a] Without proof. For a setup and measurement settings to show a violation of the inequality refer to [20]

Here the indices $o$ stand for the ordinary beam, $e$ for the extraordinary beam and $u$ for no detection. Hence, $n_{oo}(\alpha_1, \beta_1)$ indicates the count rate for both ordinary beams, while the rotation is set at angles $\alpha_1$ for $A$ and $\beta_1$ for $B$. This inequality can either be derived the same way as the CH inequality or by looking at the possible options for photon detection and only including those options compliant with the *locality* (refer to [20] for the proof). This inequality is again independent of the total number of events, just as the CH-inequality. While CHSH have then first optimized the violation of their inequality and then introduced the non-ideal efficiencies $\eta$, Eberhard [20] proposes to first introduce those efficiencies and then optimize the violation of the inequality. By doing so, the background $\zeta$ has to be taken into account. The correction takes the form of

$$\mathcal{J}_{\mathcal{B}} = \mathcal{J}_{\mathcal{B}}{}^{ideal} + 2N\zeta. \tag{57}$$

Similar to inequality 2.8, local theories now predict

$$\mathcal{J}_{\mathcal{B}} \geq 0 \tag{58}$$

Furthermore, the quantum mechanical prediction is

$$\mathcal{J}_{\mathcal{B}} = \langle\psi|\, \mathcal{B}\, |\psi\rangle \tag{59}$$

with the so called Bell operator $\mathcal{B}$, for the entangled state $|\psi\rangle$ where

$$\mathcal{B} = N\frac{\eta}{2}\begin{vmatrix} 2-\eta+\xi & 1-\eta & 1-\eta & A^*B^*-\eta \\ 1-\eta & 2-\eta+\xi & AB^*-\eta & 1-\eta \\ 1-\eta & A^*B-\eta & 2-\eta+\xi & 1-\eta \\ AB-\eta & 1-\eta & 1-\eta & 2-\eta+\xi \end{vmatrix} \tag{60}$$

here given in a basis of the states $|H,H\rangle$, $|H,V\rangle$, $|V,H\rangle$ and $|V,V\rangle$ ($H$ horizontal - , $V$ vertical polarization) (see [20]) with

$$\begin{aligned} A =& \frac{\eta}{4}(e^{2i(\alpha_1-\alpha_2)} - 1), \\ B =& \frac{\eta}{4}(e^{2i(\beta_1-\beta_2)} - 1), \\ \xi =& \frac{4\zeta}{\eta}. \end{aligned} \tag{61}$$

Now, to contradict the prediction by local theories, the expectation value of the operator $\mathcal{B}$ must be negative. This is only possible, as long as the operator $\mathcal{B}$ has a negative eigenvalue, which is limited by the background $\zeta$ [20]. This problem has been solved numerically by Eberhard [20], with the results depicted in fig. 8.

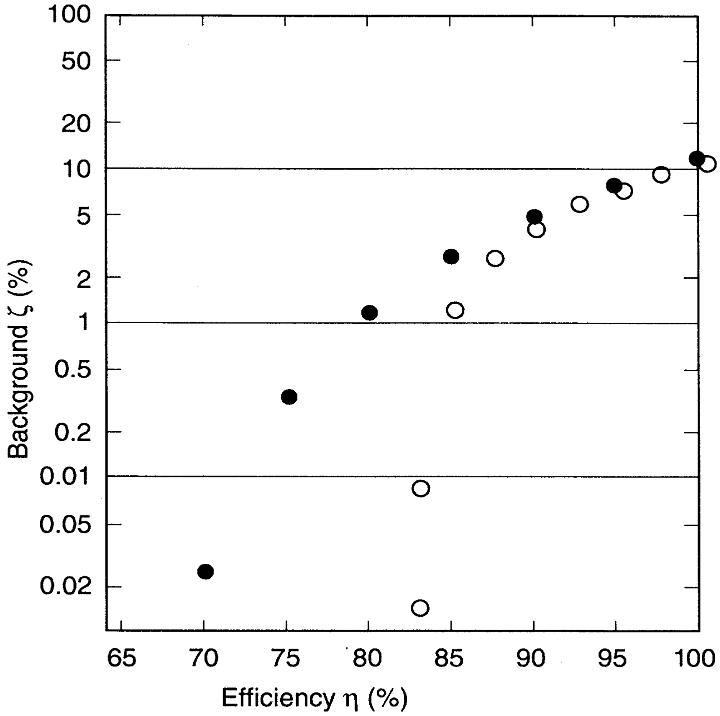



Figure 8: Maximum affordable background vs efficiency, • optimized conditions from Eberhard [20], ∘ conditions as shown by CHSH (figure from [20])

The optimal vector $|\psi\rangle$ is of the form

$$|\psi\rangle = \frac{1}{2\sqrt{1+r^2}} \begin{vmatrix} (1+r)e^{(}-i\omega) \\ -(1-r) \\ -(1-r) \\ (1+r)e^{i\omega} \end{vmatrix} \tag{62}$$

again given in a basis of the states $|H, H\rangle$, $|H, V\rangle$, $|V, H\rangle$ and $|V, V\rangle$ ($H$ horizontal - , $V$ vertical polarization) (see [20]). This can be reached by using a superposition of horizontal ($H$) and vertical ($V$) polarization, with

$$|\psi_0\rangle = \frac{1}{\sqrt{1+r^2}} \left(|H, V\rangle + r\,|V, H\rangle\right). \tag{63}$$

This state vector with the free parameter $r$ is maximally entangled for a value of $r = -1$ and a product state at $r = 0$. Depending on the background, this parameter must be optimized to gain a lower bound for the detection efficiency while still allowing for a loophole free experiment. This optimization has been performed by e.g. Giustina et al. [26] for their photon based experiment. At a measured efficiency of 76.2% and 78.6% for Alice and Bob respectively, the experiment would still be prone to the detection loophole, if the CH- or CHSH-inequalities were used.

# 3 Experiments and Loopholes

## 3.1 Overview

Based on the above given theory, many different experiments have been performed. The earlier experiments struggled with funding, as topics like the foundations of QM were quite unpopular at the time. When J. Clauser had an appointment with Richard Feynman at Caltech, to discuss a Bell test, he immediately threw him out of his office, saying: '*Well, when you have found an error in quantum theory's experimental predictions, come back then, and we can discuss your problem with it.*' [10, p48]. Still, in 1972 Clauser was able to perform his proposed experiment together with S. Freedman. They were able to show a violation of the CHSH inequality modified by Freedman. However, the interest such experiments remained quite low until the 1980s. [10, p48 ff]

Raised interest in QIT has improved the situation from the early 1990s. However, early experiments suffer from low signals such that the results are statistically not as significant. They were also subject to different loopholes, which could still allow a LHV theory to produce the observed correlations, which shall be discussed in this section. [10, p48 ff]

For those experiments, different types of entangled systems have been used. An overview is given in table 1. The list of experiments is by no means exhaustive, but shall give a short overview on some of the types of experiments performed based on Bertlmann and Zeilinger [10, p477 f] with added recent experiments and closed loopholes, such as mentioned in the cited papers [24, 34, 17, 4, 52, 55, 53, 45, 27, 54, 40, 2, 33, 25, 31, 26, 51, 57, 1, 44] and according to Larsson [39] and Myrvold, Genovese, and Shimony [41].

## 3.2 Experimental Sources of Entangled States

For the experiments in table 1, the sources of entangled states used shall be shortly described in the following subsections. Usually one distinguishes massive systems such as ions, superconducting qubits and atoms on one hand and other (non massive) systems such as photons produced by atomic cascades or PDC as well as electron spins on the other hand.

### 3.2.1 Atomic Cascade Photons

The earliest experiments such as [24, 34] on Bell's theorem measured the polarization of photons emitted in an $J = 0 \rightarrow J = 1 \rightarrow J = 0$ atomic cascade. Figure 9 shows the energy levels in the Ca atom as well as the used excitation and emission paths used by Freedman and Clauser [24] in 1972. For this experiment, they used a beam of calcium atoms emitted from an oven, which were excited to the $3d4p^1P_1$ state using a deuterium lamp. About 7% of the atoms excited this way decay to the $4p^2\ ^1S_0$ state. From there they can further decay by emitting two

Table 1: Overview of experimental tests of Bell inequalities (incomplete list) with different entangled systems, the avoided loopholes (see section 3.3) and their reported violation of the used inequality (based on [10, p477 f] with added recent experiments).

| Year | Author(s) [Ref.] | Bell Ineq. | System | closed loopholes[a] | violation |
|---|---|---|---|---|---|
| 1972 | Freedman and Clauser [24] | CHSH-F[b] | Ca cascade photons, UV excitation | | $6.3\sigma$ |
| 1973 | Holt [34] | CHSH-F[b] | Hg cascade photons, electron-beam excitation | | None |
| 1976 | Clauser [17] | CHSH-F[b] | Hg cascade photons, electron-beam excitation | | $4.1\sigma$ |
| 1982 | Aspect, Dalibard, and Roger [4] | CHSH | Ca cascade photons, laser excitation | locality | $5.1\sigma$ |
| 1988 | Shih and Alley [52] | CHSH-F[b] | parametric down conversion (PDC) photons | | $3\sigma$ |
| 1998 | Weihs et al. [55] | CHSH | Type II PDC photons | locality (> 400 m), memory, coincidence | $37\sigma$ |
| 1998 | Tittel et al. [53] | CHSH | PDC photons, Franson Energy-Time entanglement | locality (> 10 km) | $> 16\sigma$ |
| 2001 | Rowe et al. [45] | CHSH | Induced entanglement of 2 Be ions[c] | detection | $8.3\sigma$ |
| 2004 | Go [27] | CHSH | K and B meson decay (flavour entanglement) | detection | $> 3\sigma$ |
| 2007 | Ursin et al. [54] | CHSH | Type II PDC photons | locality (144 km), coincidence | $14\sigma$ |
| 2008 | Matsukevich et al. [40] | CHSH | Induced entanglement of trapped Yb ion | detection | $> 3.1\sigma$ |
| 2009 | Ansmann et al. [2] | CHSH | Induced entanglement of Josephson phase qubits | detection | $244\sigma$ |
| 2012 | Hofmann et al. [33] | CHSH | Induced entanglement of 2 trapped Rb atoms | detection | $2.1\sigma$ |
| 2013 | Giustina et al. [25] | CH-E[d] | PDC photons (non maximally entangled) | detection, coincidence | $69\sigma$ |
| 2015 | Hensen et al. [31] | CHSH | electron spin in diamond chip (event ready scheme) | 'loophole free'[e] | $p = 0.039$[f] |
| 2015 | Giustina et al. [26] | CH-E[d] | PDC photons (non maximally entangled) | 'loophole free'[e] | $11.5\sigma$ |
| 2015 | Shalm et al. [51] | CH | PDC photons (non maximally entangled) | 'loophole free'[e] | $p \leq 2.3 \cdot 10^{-7}$[f] |
| 2017 | Yin et al. [57] | CHSH | PDC photons | locality (1200 km) | $4\sigma$ |
| 2018 | Abellán et al. [1] | multiple | multiple (ions, photons, atoms, solid state) | 'loophole free'[eg] | $9.3\sigma$ to $140\sigma$ |
| 2018 | Rauch et al. [44] | CHSH | Type 0 PDC photons (non maximally entangled) | 'loophole free'[eh] | $9.3\sigma$ |

[a] As reported in the cited paper, by Larsson [39] or Myrvold, Genovese, and Shimony [41]
[b] CHSH-Freedman
[c] Spatially unresolved as both ions are located in the same ion-trap 1 $\mu$m apart.
[d] CH-Eberhart
[e] As defined by Larsson [39]
[f] Hypothesis test, p value for local-realist model producing observed data.
[g] Strong freedom of choice exclusion by using randomized settings generated from about 100,000 human participants.
[h] Strong freedom of choice exclusion for 96% of the space time volume of the past light cone of the experiment using light from distant quasars

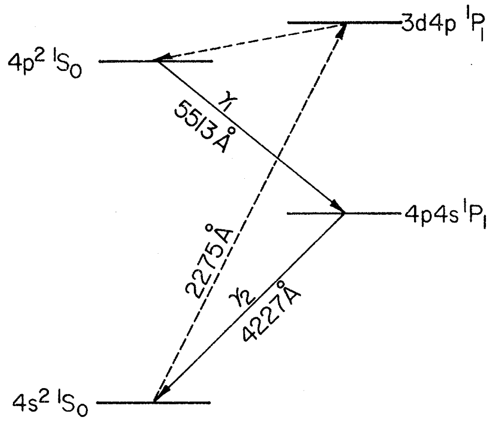



Figure 9: Level scheme of calcium. Dashed lines show the route for excitation to the initial state, solid lines show the two occuring photon emissions at wavelengths $\lambda_1 = 5513$ Å and $\lambda_2 = 4227$ Å. (Graphic from [24])

photons at wavelengths $\lambda_1 = 5513$ Å and $\lambda_2 = 4227$ Å. These photons are convenient for the experiments as they are in the visible range. Since the angular momentum and parity in the initial and final state is the same, the polarization vectors $\vec{e}_1$ and $\vec{e}_2$ show a correlation of the form $(\vec{e}_1 \cdot \vec{e}_2)^2$. For more details see Kocher and Commins [37].

### 3.2.2 Spontaneous Parametric-Down-Conversion

Later experiments using photons used the effect of parametric down conversion (PDC), first demonstrated by Burnham and Weinberg [14]. Parametric down conversion or parametric fluorescence occurs in nonlinear optical crystals. Spontaneously an incoming photon may decay into two other photons emitted simultaneously, but must not violate the conservation of momentum and energy [14], such that

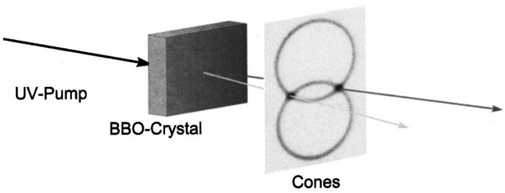
$$\vec{k}_p = \vec{k}_1 + \vec{k}_2, \tag{64}$$

$$\omega_p = \omega_1 + \omega_2. \tag{65}$$

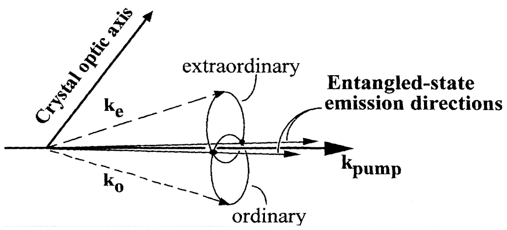
In type I down conversion the photons have the same polarization, in type II they have a different polarization [58]. A representation of the radiation as well as the principle setup is depicted in fig. 10.

UV Photons are used to pump a nonlinear crystal either in a pulsed or continuous fashion. Kwiat et al. [38] used a BBO crystal while newer experiments such as Rauch et al. [44] use periodically poled potassium titanyl phosphate (ppKTP) crystals. Of the resulting entangled states those with matching frequency (green circles in fig. 10) are used. [58]

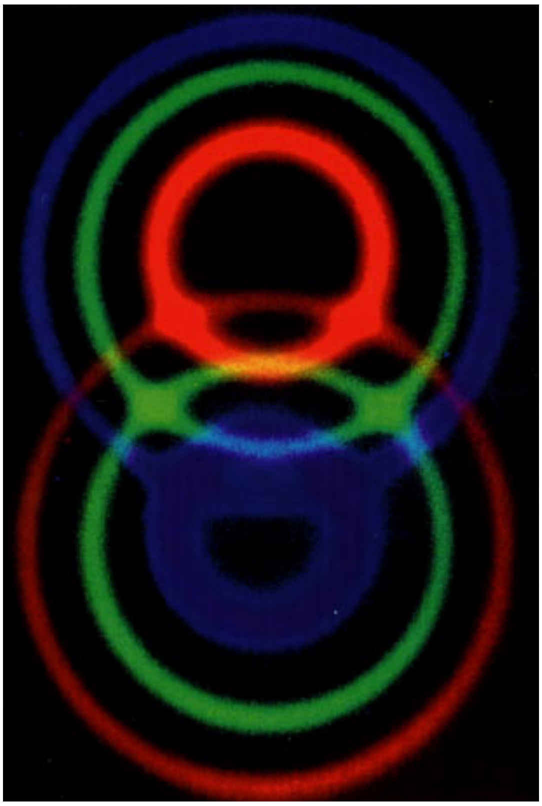
For Bell experiments with photons this technique represented a major improvement as high quality entangled states could be obtained. Furthermore, by setting the polarization of the




(a) Principle of type II PDC. The incident UV light is sent into the beta-barium-borate (BBO) crytal where they spontaniously decay into two other photons which can be found on opposite sides along the cones. From [58]




(b) Wave vectors in the photo emission relative to the optical axis of the crystal. Type II polarization entangled photons are obtained along the diections where the light cones intersect. From [38]



(c) Radiation produced by type II PDC taken with filters for different wavelength and artificial colourization for visualization. Photons are emitted with a variety of different momenta and frequencies. If, for example one photon of a pair is found on the small red circle, the other photon will be found on the small blue circle to conserve energy and momentum. From [58]

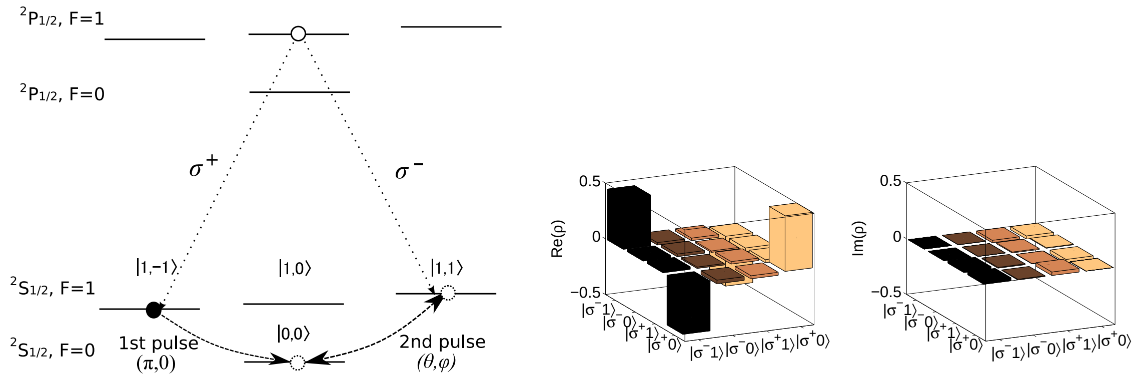
Figure 10: Principle and radiation emission for PDC

pump light source, non-maximally entangled states can be produced which are necessary to violate the CH-Eberhart inequality for lower detection rates. Together with another major improvement, superconducting nanowire single photon detectors, Giustina et al. [25] were able to close the detection loophole for photons.

### 3.2.3 Ions

A Bell experiment using ions was proposed to close the detection efficiency loophole, as massive systems such as ions allow for a high detection efficiency. [40]

To prepare an entangled state, two ions are prepared at low temperature (e.g. by laser cooling) in two Paul-Traps in different vacuum chambers. These ions are then entangled using photons. An energy level scheme for this process is shown in fig. 11




(a) Level scheme of the ion-photon entanglement. $\sigma^+$ and $\sigma^-$ indicate right and left circular polarization. From [40]

(b) Real (left) and imaginary (right) part of the density matrix produced by entangling a single photon with a single ion based on 42,495 events. From [40]

Figure 11: Principle of induced entanglement of spatially separated ions using photons as demonstrated by Matsukevich et al. [40].

At first, the ion is pumped into the $^2P_{1/2},\ F = 1\ |1, 0\rangle$ state, from where it can decay be either emitting a right ($\sigma^+$) or left ($\sigma^-$) circular polarized photon. This results in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1, 1\rangle\,|\sigma^-\rangle - |1, -1\rangle\,|\sigma^+\rangle), \tag{66}$$

since the emission of a $\sigma^+$ photon always results in a $|1, 1\rangle$ ion and equivalent for $\sigma^-$. The photons from both ions are then sent to a $\lambda/4$ wave plate to convert the circular polarization to linear polarization either horizontal ($H$) or vertical ($V$), such that $|\sigma^-\rangle \rightarrow |V\rangle$ and $|\sigma^+\rangle \rightarrow |H\rangle$. The photons are then sent to a 50/50 beam splitter (where they interfere) and then detected.

When two photons arrive at the detector (given perfect mode-matching of the input photons) the photons are detected in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|H\rangle\,|V\rangle - |V\rangle\,|H\rangle). \tag{67}$$

Due to the previous entanglement with the ions $A$ and $B$, this projects them into the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle_A\,|1,-1\rangle_B - |1,-1\rangle_A\,|1,1\rangle_B). \tag{68}$$

Now the state of the ions can be rotated into the desired basis using microwaves and measured using fluorescence with high efficiency. For more details see Matsukevich et al. [40].

Before highly efficient photon detectors were developed, a combined system of ions and photons was proposed to close both the detection and locality loophole. A first experiment with ions was carried out by Rowe et al. [45], however, both ions were still held in the same trap. Matsukevich et al. [40] improved on this result and could show a violation of a Bell inequality for ions in two different traps about 1 m apart. [40, 41]

Newer experiments, e.g. Häffner et al. [30] were able to demonstrate scalable multi particle entanglement for ions.

### 3.2.4 Other Sources

A variety of other systems has been used for Bell tests, such as solid state qubits [2], electron spin in nitrogen vacancy (NV) [32, 31], K and B mesons [50, 27], atom clouds [33] and others.

Massive systems (ions, atoms, electron spin, solid state) have the advantage of high detection efficiency and due to the possibility of induced entanglement via photons as described above are able to meet the locality requirement. However, these systems only have a low event rate compared to systems using photons. For example, the experiment of Matsukevich et al. [40] reported a event rate of 1 entanglement every 39 s. In the 'loophole free' test from Hensen et al. [31], 245 trials over a period of 220 h was observed. In comparison, the experiment of Giustina et al. [26] reported 3500 down conversion pairs produced per second.

## 3.3 Loopholes

Due to experimental constraints, experiments must make additional assumptions to test Bell's theorem. These assumptions lead to loopholes an intricately designed local hidden variable (LHV) theory could exploit (showing a violation of Bell's inequality), while still being compatible with the assumptions used to derive Bells theorem. Every experiment is forced to make some assumptions. However, it must be mentioned that not every assumption leads to a loophole. For example, in massive systems (atoms, ions, ...) the fair sampling assumption does not lead to the detection loophole, since a high detection efficiency is inherent in the system used. [10, p488 ff]

Larsson [39] has proposed a minimal set of (untestable) assumptions which an experiment can make in order to be called loophole free. In 2015 three experiments by Hensen et al. [31], Shalm et al. [51] and Giustina et al. [26] used these criteria and are therefore called loophole free experiments. The following shall give a description of those loopholes and how to avoid them based on Bertlmann and Zeilinger [10, p488 ff], Myrvold, Genovese, and Shimony [41] and Larsson [39].

### 3.3.1 Locality or Communication Loophole

In order to test a Bell inequality, every experiment must assume locality (see definition 2.2). In other words, locality assumes whatever happens on Bobs side does not influence Alice's result in any way and vice versa. For a specific experiment one assumes 'outcome independence' and 'setting independence'. This means, neither settings nor measurement result should, if locality is 'real', be able to propagate between the measurement stations. Since this propagation of information can, in a local theory obeying the principles of relativity, only happen at the speed of light this loophole is closed by a spacelike separation of the two measurement facilities. This spacelike separation is shown in the following Minkowsky diagram from Hensen et al. [31]. As can be seen in fig. 12, information about the measurement or settings propagating from Alice (green) at the speed of light can only reach Bob (blue) after his measurement is completed. This closes the locality loophole.

Many different experiments since have closed this loophole starting with Aspect, Grangier, and Roger [5]. Due to experimental and technological advancements, increasing distances could be achieved, starting from 400 m by Weihs et al. [55], 10 km by Tittel et al. [53], 144 km by Scheidl et al. [46] up to recently 1200 km by Yin et al. [57] using a satellite based system. Zeilinger [58] proposes even bigger distances of a few light seconds in order to have humans (see freedom of choice loophole) choose the measurement setting in spacelike separation.

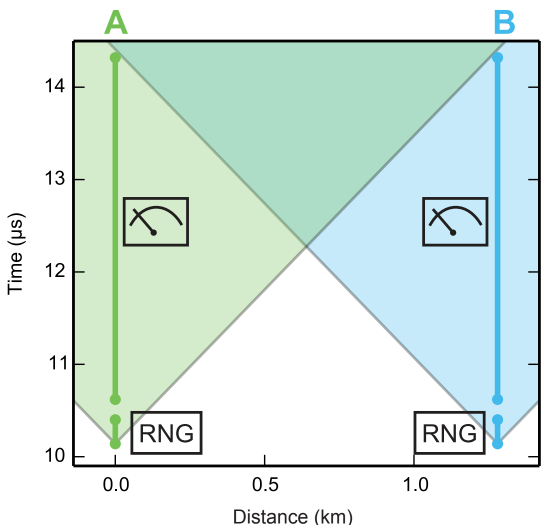



Figure 12: Minkowsky diagram showing spacelike separation of measurement settings and outcomes of Alice (A) and Bob (B) (diagram from Hensen et al. [31])

### 3.3.2 Memory Loophole

Two variants of the memory loophole exist. Aspect [3] used periodic experimental settings. He notes:

> ... a supplementary assumption should be exhibited: The polarizers have no "memory," i.e., they can be influenced by signals received at a certain time from the [remote site] (with a certain delay) but they cannot store all this information for a long time and extrapolate in the future even if there is some regularity in the [settings]. [3]

This means, if a LHV model could memorize the periodicity of the settings it could produce the observed results. This variant loophole is closed by using randomized measurement settings, see e.g. Weihs et al. [55].

The second variant of the memory loophole is based on the statistical analysis. In many experiments it is assumed that the different experimental runs are independent. However, the results of previous experiments might enter into the measurement process and determine the outcome. This loophole can be closed by using statistical analysis which does not use this independence assumption (for more details see [39, 12]).

### 3.3.3 Freedom of Choice and Superdeterminism

Freedom of choice assumes that the measurement settings are independent of any hidden variables. This is linked to the reality assumption, since reality assumes independence in the other direction. A theory obeying the reality assumption must describe the reality independent of the measurement. Since the measurement settings are free parameters, the description and therefore the hidden variables (describing element) must not depend on the measurement settings in a *real* theory. Formally, $\lambda$ should be independent of the local measurement settings. As Larsson [39] states: '*... what really matters is that the hidden variable and the local parameters do not share a common cause.*'. This assumption creates a loophole, since the whole set up, settings and the measurement outcome could be predetermined by this common cause.

This loophole is usually addressed by using sophisticated systems for random setting choice generation [39]. This setting choice is usually used to switch the path between two detectors with different settings or in case of polarized photons using e.g. a Plockels cell to rotate the polarization of the incoming photon before it is sent to the detector with a fixed polariser installed [51].

However, it can still be debated that this random choice based on physical sources of randomness could be controlled by hidden variables, ruling out this approach. As suggested by Zeilinger [58], human experimenters could, if spacelike separated by a few lightseconds, exert their (hypothetical) free will. However, as Larsson [39] mentions, a human subject is typically not a good source of randomness. With the idea of human free will, 'The BIG Bell Test Collaboration'[1] performed 13 simultaneous Bell-Experiments based on different entangled systems with randomized settings generated by about 100,000 participants playing an online video game. For more information see Abellán et al. [1].

Another option proposed by Zeilinger (see [44, 39]) would be to use experiment settings determined by the radiation from distant quasars. An experiment based on this option has been carried out by Rauch et al. [44] in 2018. Based on this experiment, any hidden variable exploiting the freedom of choice loophole must have it's cause at last $7.8 \cdot 10^9$ years ago, influencing all events leading to the experiment [44]. But, as Larsson [39] states:

> '*Even remote quasar emissions must be assumed not to have a common cause. It is possible that all events in the universe share common causes, a philosophical view called superdeterminism. ... This constitutes a loophole, but if superdeterminism holds, there is no point in discussing what mathematical models could be used to model nature, be it local realist or quantum or any other model. ... The loophole of superdeterminism cannot be closed by scientific methods; the assumption that the world is not superdeterministic is needed to do science in the first place.*' [39]

This means, even if superdeterminism governs our universe and everything is defined starting from an event S, we could not perform any experiment *inside* this universe in order to test this idea, since even the idea itself, as well as the outcome of the experiment would already be predetermined.

#### 3.3.4 Efficiency or Detection Loophole

This loophole exploits the fair sampling assumption. Due to experimental limitations - loss of entangled pairs in the setup or low detector efficiency - any experiment must assume that the sample taken resembles the whole ensemble. But what if only states are detected which violate the inequality to be tested but the whole ensemble does not? It can be shown that a local hidden variable controlling detection can violate a Bell inequality. [39]

One approach is to use an inequality free of the fair sampling assumption, such as the CH- or CH-Eberhart-inequality (see [18, 20]). However, these inequalities require in themselves high detection efficiencies ($> 2/3$ of all events for the CH-Eberhart-inequality), such that experiments using those appeared only recently, see e.g. [25].

Another approach is to ensure output data on every experimental run. The first experiment to implement this feature is Rowe et al. [45], using trapped ions 3 $\mu$m apart. Still, those ions are still held in the same ion-trap, opening the locality loophole. Later Matsukevich et al. [40] improved on this result with ions held in separate traps 1 m apart. A different improvements was made by Ansmann et al. [2], where a superconducting system of Josephson phase qubits was used. Based on the high detection efficiency and an average of 34.1 million runs they obtained a Bell violation of the CHSH inequality of $244\sigma$. However, the two qubits are only separated by 3.1 mm, which again does not close the locality loophole.

The third approach that shall be mentioned here, employs a 'heralded' source, such as proposed by Clauser and Shimony [16] and used by Hofmann et al. [33] or Hensen et al. [31] (here called event ready scheme). Instead of a source sharing an already entangled state, the process employed entangles the two systems at Alice's and Bob's location and produces a signal at the source if the process was successful. For further details see [59, 33, 31].

Other approaches exist theoretically, such as using different systems for each of the two sites (e.g. an ionic system at A, a photonic at B), where a high efficiency on one site allows for a lower efficiency on the other site. See Larsson [39] for further details.

#### 3.3.5 Coincidence Loophole

Many experiments, especially the ones performed with photons, rely on the detection of a pair of particles. Here, the identification of pairs is especially problematic with rapid repetition setups, using e.g. PDC. In order to identify a pair of particles, the relative timing of the detection at the sites is often used. Detections relatively close in time are assigned to the same pair, but could also belong to separate pairs. This loophole is problematic because it concerns a joint property - two particles form a pair or not.

Here again an event ready setup can be used, as pair detection is linked to the time of the event ready signal. Any detections which do not coincide with the event ready signal can be attributed to the detector efficiency and analysed as such.

A simpler option is to use fixed time slots for the detection. Again, time slots where no detection or a single sided detection occurred can be accounted to to the detector efficiency. As pulse pumped experiments already use this technique, they are not vulnerable to this loophole.

For more details and a derivation of a specialized CH-like inequality see Larsson [39].

### 3.3.6 Other Loopholes

Dependent on the specific setup and assumptions, experiments may be vulnerable to variety of other different loopholes, some of which may be categorized as a variation of those already described. Others, like superdeterminism have no experimental significance, as they can not be tested. [10, p492 f]

Another loophole should be mentioned here, introduced by Kent [35] as the 'collapse locality loophole'. It is proposed that the QM state does not collapse immediately when measured and recorded, resulting in a violation of the locality criterion. As the state would not yet be collapsed, information may be exchanged violating the locality assumption. However, it is still unclear when such a collapse should take place; it is argued that interaction with sufficient gravity-fields or a 'concious' observer lead to such a collapse, until which the two sites must remain at spacelike separation. As the observation by a 'concious' observer (human) would require a spacelike separation of light seconds, this could not yet be experimentally addressed. For more details see Kent [35, 36].

## 3.4 Loophole Free Experiments

In 2015 three papers were published claiming 'loophole free' tests of bell inequalities in the sense of Larsson [39]. Here, a short summary of two of those experiments will be given, while the third is described in more detail in the next section.

The first experiment from Hensen et al. [31] is based on the electron spin associated with a single nitrogen vacancy (NV) in a diamond chip. This experiment also uses an event ready scheme. Both NV spins are at first entangled with the emission time of a single photon each (time bin encoding). '*The two photons are then sent to location C, where they are overlapped on a beam-splitter and subsequently detected. If the photons are indistinguishable in all degrees of freedom, the observation of one early and one late photon in different output ports projects the spins at A and B into the maximally entangled state.*' [31] Only in this case, the bell experiment is performed (event ready), excluding all cases where the entanglement distribution failed. However, this experiment only included 245 trials over a measurement time of 220 h, thus yielding only a small statistical significance of the CHSH-inequality (theorem 2.4) violation of $S_{CHSH} = 2.42 \pm 0.20$. [31, 41]

The two other experiments by Shalm et al. [51] and Giustina et al. [26] are both based on polarized photons. A more in depth description of the Giustina et al. [26] experiment will be given in the next subsection.

Shalm et al. [51] uses a pulsed Ti:Sapphire laser for exciting the ppKTP crystal (spontaneous PDC) to produce entanglement optimized photon pairs (non maximally entangled). These pairs are then distributed to measurement stations A and B. While the photons are in flight, the settings are chosen. At each station the photon is sent through a Pockels cell and a polariser to perform the polarization measurement based on the chosen settings. Then the photon is detected by a superconducting nanowire single-photon detector with an efficiency of $(91 \pm 2)\%$. The overall reported system efficiency is $> 74\%$, which satisfies the criteria for the detection loophole set by [20] and the analysis of the background count of $> 72.5\%$. They use the CH inequality (theorem 2.6) with additional test statistics (see [51] for more details) and report a smallest adjusted p-value[4] of $2.3 \cdot 10^{-7}$ based on 12,127 trials over 30 min. [51, 41]

[4] p-value for the null hypothesis that the measured probabilities in the experiment are constrained by local realism [51].

## 3.5 Photon based Experiment by Giustina et. al. 2015

The experiment performed by Giustina et al. [26] was one of the first experiments to close all major loopholes in a Bell test. They used a PDC source (a ppKTP crystal), electrically randomized settings at spatial separation as well as highly efficient detectors. This, together with the according careful data analysis, employing the CH-Eberhard inequality (see section 2.5) with statistical improvements by Bierhorst [12] and optimized entangled states allowed to close the locality, freedom of choice and detection loopholes.

An overview of the experimental setup is shown in fig. 13.

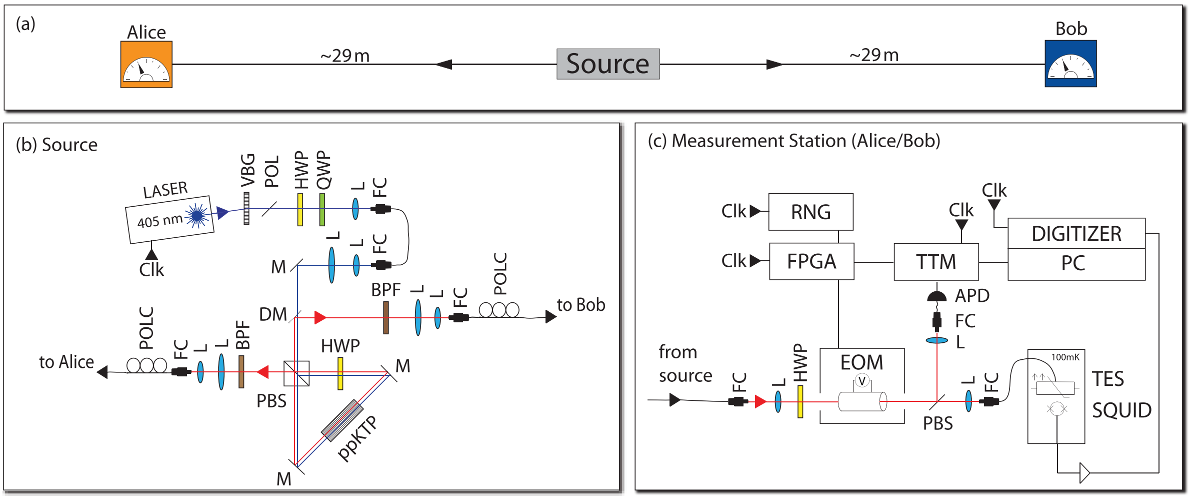



Figure 13: (a) Schematic of the experimental setup. (b) Source: The source distributes two polarization entangled photons to the measurement stations Alice and Bob. The photons are generated using type II PDC in a ppKTP crystal pumped with a 405 nm laser. Each pair is split using a polarizing beam splitter (PBS) and sent via two single-mode fibres to the stations. (c) Measurement Stations: In each of the stations a polarization setting is is chosen via the random number generator (RNG) and the electro optical modulator (EOM), a Plockels cell, set to the according polarization. The such modified photon is then detected by the transition edge sensor (TES) single photon detector. Abbreviations: APD, avalanche photodiode; BPF, bandpass filter; DM, dichroic mirror; FC, fiber connector; HWP, half-wave plate; L, lens; POL, polarizer; M, mirror; POLC, manual polarization controller; QWP, quarter-wave plate; SQUID, superconducting quantum interference device; TES, transition-edge sensor; TTM, time-tagging module. From [26]

The setup was located in the Vienna Hofburg castle, with a distance of 29 m between the source and either measurement station, called 'Alice' and 'Bob'. To ensure synchronous timing, the whole setup was clocked by a 10 MHz master oscillator, with the laser pumping the ppKTP crystal and the Plockels cells switched at a rate of 1 MHz. The Plockels cell at each station changed the polarization depending on the locally generated random settings before the photon reached the polarizer, therefore selecting one of two possible settings. The timings in this experiment were carefully characterized to ensure spatial separation, not only of the measurement of the photons but also of the setting generation [26]. A space-time diagram is shown in fig. 14.

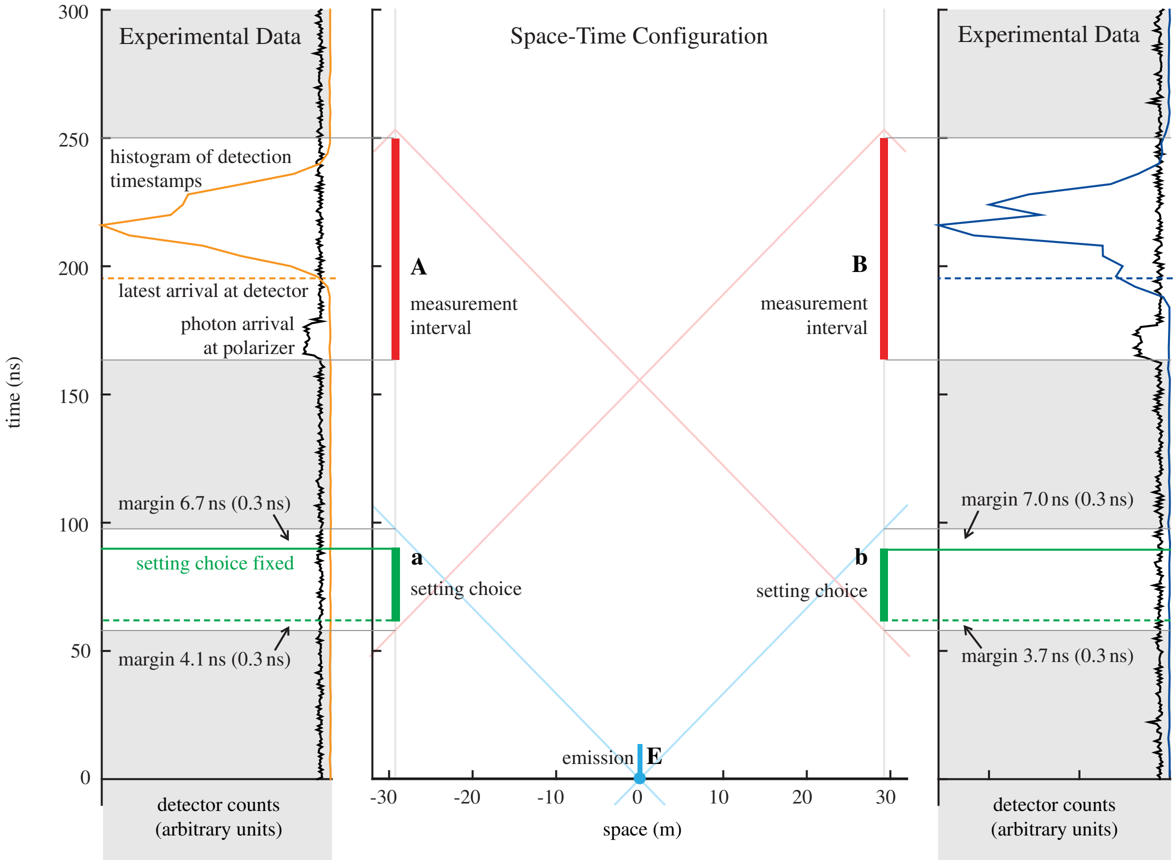



Figure 14: Spacetime configuration of the experiment. The center depicts the spacetime configuration of the experiment, where the emission interval (E) is represented by the solid blue line, the setting generation interval is represented by the green line (a,b) and the measurement interval is marked in red (A,B). The setting choice generation is constrained by the backward light cone of the measurement interval as well as the forward lightcone of the emission intervall. The blocks on the left and right depict the experimental data from Alice and Bob respectively. The orange and blue lines depict histograms of the photon detection times, where the dotted lines mark the latest possible arrival time at the TES. The black histograms depict the arrival time of photons at the plate PBS. (from [26])

With the setting choice generated *before* the photons can reach the station as well as the measurement at the other station completed before the setting choice can possibly reach it, space-like separation is ensured. For the analysis, the CH-Eberhard inequality of the form

$$p_{++}(a_1, b_1) - p_{+0}(a_1, b_2) - p_{0+}(a_2, b_1) - p_{++}(a_2, b_2) \leq 0 \tag{69}$$

was employed, where + indicates detection and 0 indicates no detection with the settings $a_1$ and $a_2$ at Alice and $b_1$ and $b_2$ at Bob. The state produced by the source is of the form

$$|\psi\rangle = \frac{1}{\sqrt{1+r^2}}(|V\rangle_A |H\rangle_B + r |H\rangle_A |V\rangle_B) \tag{70}$$

with $H$ for the horizontal and $V$ for the vertical polarization. The system was characterized with both the product state ($r = 0$) as well as the maximally entangled state ($r = -1$). Efficiencies of 76.2% and 78.6% were be measured at Alice and Bob respectively. The efficiency represents the the ratio of the coincident counts divided by the total number of single counts. The experiment was then performed at a setting of $r \approx -2.9$ at angles of $a_1 = 94.4^o$, $a_2 = 62.4^o$, $b_1 = -6.5^o$ and $b_2 = 25.5^o$. The observed violation of inequality (69) is depicted in fig. 15.

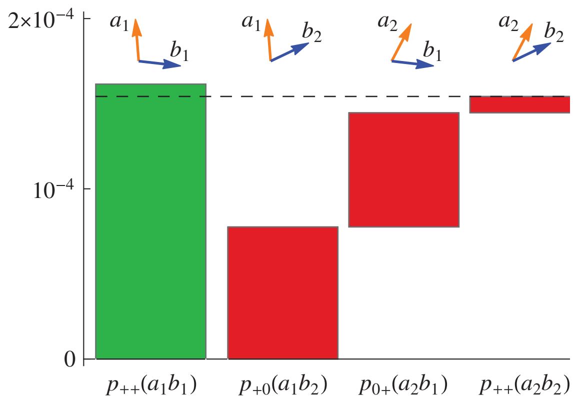



Figure 15: Violation of the CH-Eberhard inequality as observed by Giustina et al. [26]. The green bar outweighs the sum of the other red bars, violating the inequality. From [26]

With this result, Giustina et al. [26] report, given their large sample size of approx. $12.3 \cdot 10^6$ samples taken in 3510 seconds (58.5 min), a violation of 11.5 standard deviations of the employed inequality.

Although the results from this experiment are conclusive, further experiments have been proposed and performed, especially to significantly improve on the freedom of choice loophole.

## 3.6 Further Developments

After these experiments closing the most significant loopholes simultaneously, proposals to improve the 'closure' of the freedom of choice loophole as well as other improvements have been realized.

The BIG Bell Test Collaboration [1] performed 13 different experiments in 12 Laboratories across 5 continents during a 12 h period. In order to close the freedom of choice loophole they used settings provided by about 100,000 human participants playing an online game. The performed experiments were conclusive with a resulting significance of up to $140\sigma$. For more details see [1].

Rauch et al. [44] performed an experiment with the setting choices dependent on the light of distant quasars on opposite ends of the universe. This allowed for a strong freedom of choice exclusion in the backwards lightcone of the experiment with a statistical significance of $9.3\sigma$.

Bell tests are closely linked with quantum cryptography setups, as Bell inequalities can be used to test the quality of the shared entangled states. For quantum cryptography especially distribution over long distances is difficult. Here, Yin et al. [57] could demonstrate the distribution of an entangled state over 1200 km to a satellite via the violation of a Bell inequality.

# 4 Summary and Conclusion

In 1935, contributing to previous discussion, Einstein, Podolsky and Rosen (EPR) published their famous paper on the EPR-paradox. This '*was advanced as an argument that QM could not be a complete theory, but should be supplemented by additional variables*'[7].

Based on the assumptions of *reality* ('*without in any way disturbing a system, we can predict with certainty the value of a physical quantity*'[21]) and *locality* (all effects must have local causes, limited by the speed of light) they showed, by employing properties of entangled states, that quantum mechanics is incomplete in the sense that it can not describe specific measurement outcomes.

One possibility to complete it is by hidden variables (HVs). An example of such a theory, developed by de'Broglie and later Bohm are the Bohmian Mechanics [28]. The wave function is supplemented with additional coordinates, allowing for a remedy of the measurement problem. One subclass of HV theories are local hidden variable (LHV) theories, where the hidden variables are *local*.

Later, in 1964, based on the same assumptions as made by EPR, Bell derived his famous inequality (see theorem 2.3 for the derivation) [7]

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \leq 1 + P(\vec{b},\vec{c}) \tag{71}$$

which is violated by certain QM phenomena for the expectation value $P(\vec{a},\vec{b})$ with the measurement settings $\vec{a}$ and $\vec{b}$. Hence he could show, that the assumptions made by EPR are incompatible with QM. It must be noted here that the formulation of the assumptions has changed over the years. This is especially confusing as different sources use the same terminology with different underlying definitions (see [10, p122 ff] for a discussion and [41] for a different point of view).

Based on Bells work, CHSH could derive a slightly different inequality, better suited for the experiment. Bells original derivation relied on perfect anti-correlation of the measurement results, which is impossible in an imperfect experiment. Based on this inequality the first experiment could be performed by Freedman and Clauser [24] in 1972. This experiment showed a violation of the CHSH-inequality of $6.3\sigma$, resulting in the conclusion that nature itself is incompatible with the assumptions of *locality* and *reality* proposed by EPR.

However, early Bell experiments needed additional assumptions in order to violate the inequality. These assumptions allow that a theory still satisfying *locality* and *reality* could produce the results observed, resulting in so called 'loopholes'. One of the assumptions of the CHSH-inequality is that the total number of events is known, which is needed to compute the violation of the inequality. This problem, one part of the so called 'detection loophole' was solved by Clauser and Horne [18], where the inequality (seetheorem 2.6 for the derivation)

$$-N \leq N_{12}(a,b) - N_{12}(a,b') + N_{12}(a',b) + N_{12}(a',b') - N_1(a') - N_2(b) \leq 0, \tag{72}$$

was derived. As the normalization appears in all terms it can simply be cancelled. To violate this inequality without opening another type of detection loophole, a high detection efficiency ($> 82.8\%$) must be ensured. Eberhard [20] was able to relax this detection efficiency to $> 2/3$ for experiments with low background (see section 2.5). Other approaches to this loophole exist, such as heralding systems where a successful entanglement of a pair of particles generates a ready signal. For most experiments, other loopholes exist. The most important ones apart from the detection loophole are the locality or communication, memory loophole and freedom of choice loophole.

To close the locality loophole experimentally, the experiment must be designed such that no communication of measurement settings or measurement result can be communicated between the two stations while the measurement is still ongoing. This loophole can be closed by spatially separating the two stations and characterizing the precise timing of the setup.

The memory loophole exists in two variants, one exists for periodic settings, where it must be assumed that the polarizers have no memory, which can be closed by using randomized settings. For most of the more recent experiments the second variant is more important, where the measurement outcome could be influenced by previous outcomes. This loophole can be closed by using a statistical analysis which does not assume independent samples (see e.g. [39, 12]).

More difficult to access is the freedom of choice loophole. Every experiment must assume that some parameters, the measurement settings, can be chosen freely. However, it is unclear if this freedom of choice exist. For example, random number generators at both stations could be influenced by a cause in their common past. Different approaches have been proposed and experimentally tested. Zeilinger [58] proposed to use radiation from distant quasars or human participants at spacelike separation large enough to produce the measurement settings. The first approach has been tested by Rauch et al. [44], the second is at the moment limited by the accessible spatial distance, but Abellán et al. [1] has shown several Bell violations based on measurement settings generated from 100.000 participants.

Since Bell published his inequality, many different experiments based on different systems have been proposed and performed. An (incomplete) overview can be seen in table 1, where the employed inequality, entangled system and the closed loopholes as well as the statistical significance of the violations are listed.

Especially noteworthy are the first so called 'loophole free' experiments (in the sense of Larsson [39]) in 2015 by Hensen et al. [31], Giustina et al. [26] and Shalm et al. [51]. These experiments were the first to close all of the above mentioned major loopholes in a single experiment. Hensen et al. [31] employed a nitrogen vacancy (NV) in a diamond chip and an event ready scheme. Due to the low event rate of the system of less than one event per hour only 245 trials could be performed. Photonic systems, such as employed by Giustina et al. [26] and Shalm et al. [51], generate events at a much higher rate, in the case of Giustina et al. [26] approx. 3500 photon pairs per second.

Further improvements on these experimental results have been made by Rauch et al. [44] (strong freedom of choice exclusion by using settings generated from radiation from distant quasars), Yin et al. [57] (entanglement distribution over 1200 km) and Abellán et al. [1] where data from human participants was used in 13 different Bell experiments across 5 continents during a 12 h period.

It can be concluded that the experimental evidence collected shows a strong contradiction between quantum mechanics and the assumptions of *locality* and *reality*. Of these assumptions it is unclear and so far not experimentally testable which to reject. Several different viewpoints exist on this topic (see [10, p122 ff] or [41] for a discussion). Additionally it must be said that one could also reject the experimental evidence by rejecting the still experimentally existing auxiliary assumptions. If for example everything is predetermined (superdeterminism), the experiment could not decide if it was so. Furthermore there exist objections around the QM-collapse by Kent [36]. If we assume that nature does not conspire against our experiment, we can reject certain theories (LHV) as a completion of quantum mechanics.

# 5 References

[1] C. Abellán et al. "Challenging Local Realism with Human Choices". In: *Nature* 557.7704 (May 2018), pp. 212–216. ISSN: 0028-0836, 1476-4687. DOI: 10.1038/s41586-018-0085-3. URL: http://www.nature.com/articles/s41586-018-0085-3 (visited on 01/25/2020).

[2] Markus Ansmann et al. "Violation of Bell's Inequality in Josephson Phase Qubits". In: *Nature* 461.7263 (Sept. 2009), pp. 504–506. ISSN: 1476-4687. DOI: 10.1038/nature08363. URL: https://www.nature.com/articles/nature08363 (visited on 01/11/2020).

[3] Alain Aspect. "Proposed Experiment to Test the Nonseparability of Quantum Mechanics". In: *Physical Review D* 14.8 (Oct. 15, 1976), pp. 1944–1951. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.14.1944. URL: https://link.aps.org/doi/10.1103/PhysRevD.14.1944 (visited on 01/13/2020).

[4] Alain Aspect, Jean Dalibard, and Gérard Roger. "Experimental Test of Bell's Inequalities Using Time- Varying Analyzers". In: *Physical Review Letters* 49.25 (Dec. 20, 1982), pp. 1804–1807. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.49.1804. URL: https://link.aps.org/doi/10.1103/PhysRevLett.49.1804 (visited on 01/12/2020).

[5] Alain Aspect, Philippe Grangier, and Gérard Roger. "Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities". In: *Physical Review Letters* 49.2 (July 12, 1982), pp. 91–94. DOI: 10.1103/PhysRevLett.49.91. URL: https://link.aps.org/doi/10.1103/PhysRevLett.49.91 (visited on 01/11/2020).

[6] J. S. Bell. "Bertlmann's Socks and the Nature of Reality". In: *Le Journal de Physique Colloques* 42.C2 (Mar. 1981), pp. C2-41-C2–62. ISSN: 0449-1947. DOI: 10.1051/jphyscol:1981202. URL: http://www.edpsciences.org/10.1051/jphyscol:1981202 (visited on 09/12/2019).

[7] J. S. Bell. "On the Einstein Podolsky Rosen Paradox". In: *Physics Physique Fizika* 1.3 (Nov. 1, 1964), pp. 195–200. ISSN: 0554-128X. DOI: 10.1103/PhysicsPhysiqueFizika.1.195. URL: https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195 (visited on 09/12/2019).

[8] J. S. Bell. *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*. Cambridge [Cambridgeshire] ; New York: Cambridge University Press, 1987. 212 pp. ISBN: 978-0-521-33495-2.

[9] Mary Bell and Shan Gao, eds. *Quantum Nonlocality and Reality: 50 Years of Bell's Theorem*. Cambridge: Cambridge University Press, 2016. ISBN: 978-1-316-21939-3. DOI: 10.1017/CBO9781316219393. URL: http://ebooks.cambridge.org/ref/id/CBO9781316219393 (visited on 09/12/2019).

[10] Reinhold Bertlmann and Anton Zeilinger, eds. *Quantum [Un]Speakables II: Half a Century of Bell's Theorem*. The Frontiers Collection. Cham: Springer International Publishing, 2017. ISBN: 978-3-319-38985-1 978-3-319-38987-5. DOI: 10.1007/978-3-319-38987-5. URL: http://link.springer.com/10.1007/978-3-319-38987-5 (visited on 01/11/2020).

[11] Reinhold A. Bertlmann. "Bell's Universe: A Personal Recollection". In: (May 25, 2016). arXiv: 1605.08081 [physics, physics:quant-ph]. URL: http://arxiv.org/abs/1605.08081 (visited on 01/05/2020).

[12] Peter Bierhorst. "A Robust Mathematical Model for a Loophole-Free Clauser–Horne Experiment". In: *Journal of Physics A: Mathematical and Theoretical* 48.19 (May 15, 2015), p. 195302. ISSN: 1751-8113, 1751-8121. DOI: 10.1088/1751-8113/48/19/195302. URL: https://iopscience.iop.org/article/10.1088/1751-8113/48/19/195302 (visited on 07/18/2020).

[13] David Bohm. *Quantum Theory*. New York: Dover Publications, 1989. 646 pp. ISBN: 978-0-486-65969-5.

[14] David C. Burnham and Donald L. Weinberg. "Observation of Simultaneity in Parametric Production of Optical Photon Pairs". In: *Physical Review Letters* 25.2 (July 13, 1970), pp. 84–87. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.25.84. URL: https://link.aps.org/doi/10.1103/PhysRevLett.25.84 (visited on 01/11/2020).

[15] H. A. Carteret, A. Higuchi, and A. Sudbery. "Multipartite Generalization of the Schmidt Decomposition". In: *Journal of Mathematical Physics* 41.12 (Dec. 2000), pp. 7932–7939. ISSN: 0022-2488, 1089-7658. DOI: 10.1063/1.1319516. URL: http://aip.scitation.org/doi/10.1063/1.1319516 (visited on 01/24/2020).

[16] J F Clauser and A Shimony. "Bell's Theorem. Experimental Tests and Implications". In: *Reports on Progress in Physics* 41.12 (Dec. 1, 1978), pp. 1881–1927. ISSN: 0034-4885, 1361-6633. DOI: 10.1088/0034-4885/41/12/002. URL: https://iopscience.iop.org/article/10.1088/0034-4885/41/12/002 (visited on 06/24/2020).

[17] John F. Clauser. "Experimental Investigation of a Polarization Correlation Anomaly". In: *Physical Review Letters* 36.21 (May 24, 1976), pp. 1223–1226. DOI: 10.1103/PhysRevLett.36.1223. URL: https://link.aps.org/doi/10.1103/PhysRevLett.36.1223 (visited on 01/12/2020).

[18] John F. Clauser and Michael A. Horne. "Experimental Consequences of Objective Local Theories". In: *Physical Review D* 10.2 (July 15, 1974), pp. 526–535. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.10.526. URL: https://link.aps.org/doi/10.1103/PhysRevD.10.526 (visited on 09/12/2019).

[19] John F. Clauser et al. "Proposed Experiment to Test Local Hidden-Variable Theories". In: *Physical Review Letters* 23.15 (Oct. 13, 1969), pp. 880–884. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.23.880. URL: https://link.aps.org/doi/10.1103/PhysRevLett.23.880 (visited on 09/12/2019).

[20] Philippe H. Eberhard. "Background Level and Counter Efficiencies Required for a Loophole-Free Einstein-Podolsky-Rosen Experiment". In: *Physical Review A* 47.2 (Feb. 1, 1993), R747–R750. DOI: 10.1103/PhysRevA.47.R747. URL: https://link.aps.org/doi/10.1103/PhysRevA.47.R747 (visited on 01/11/2020).

[21] A. Einstein, B. Podolsky, and N. Rosen. "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" In: *Physical Review* 47.10 (May 15, 1935), pp. 777–780. ISSN: 0031-899X. DOI: 10.1103/PhysRev.47.777. URL: https://link.aps.org/doi/10.1103/PhysRev.47.777 (visited on 09/12/2019).

[22] Artur Ekert and Peter L. Knight. "Entangled Quantum Systems and the Schmidt Decomposition". In: *American Journal of Physics* 63.5 (May 1995), pp. 415–423. ISSN: 0002-9505, 1943-2909. DOI: 10.1119/1.17904. URL: http://aapt.scitation.org/doi/10.1119/1.17904 (visited on 01/24/2020).

[23] Hans Gerd Evertz. *Vorlesungsskript Quantenmechanik*. Feb. 2017. URL: https://itp.tugraz.at/~evertz/QM/qm_2017.pdf (visited on 03/31/2019).

[24] Stuart J. Freedman and John F. Clauser. "Experimental Test of Local Hidden-Variable Theories". In: *Physical Review Letters* 28.14 (Apr. 3, 1972), pp. 938–941. DOI: 10.1103/PhysRevLett.28.938. URL: https://link.aps.org/doi/10.1103/PhysRevLett.28.938 (visited on 01/11/2020).

[25] Marissa Giustina et al. "Bell Violation Using Entangled Photons without the Fair-Sampling Assumption". In: *Nature* 497.7448 (May 2013), pp. 227–230. ISSN: 1476-4687. DOI: 10.1038/nature12012. URL: https://www.nature.com/articles/nature12012 (visited on 01/12/2020).

[26] Marissa Giustina et al. "Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons". In: *Physical Review Letters* 115.25 (Dec. 16, 2015), p. 250401. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.115.250401. URL: https://link.aps.org/doi/10.1103/PhysRevLett.115.250401 (visited on 09/12/2019).

[27] Apollo Go. "Observation of Bell Inequality Violation in B Mesons". In: *Journal of Modern Optics* 51.6-7 (Apr. 2004), pp. 991–998. ISSN: 0950-0340, 1362-3044. DOI: 10.1080/09500340408233614. URL: http://www.tandfonline.com/doi/abs/10.1080/09500340408233614 (visited on 01/11/2020).

[28] Sheldon Goldstein. "Bohmian Mechanics". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Summer 2017. Metaphysics Research Lab, Stanford University, 2017. URL: https://plato.stanford.edu/archives/sum2017/entries/qm-bohm/.

[29] Daniel M. Greenberger, Michael A. Horne, and Anton Zeilinger. "Going Beyond Bell's Theorem". In: (Dec. 6, 2007). arXiv: 0712.0921 [quant-ph]. URL: http://arxiv.org/abs/0712.0921 (visited on 01/11/2020).

[30] H. Häffner et al. "Scalable Multiparticle Entanglement of Trapped Ions". In: *Nature* 438.7068 (Dec. 2005), pp. 643–646. ISSN: 0028-0836, 1476-4687. DOI: 10.1038/nature04279. URL: http://www.nature.com/articles/nature04279 (visited on 01/25/2020).

[31] B. Hensen et al. "Loophole-Free Bell Inequality Violation Using Electron Spins Separated by 1.3 Kilometres". In: *Nature* 526.7575 (Oct. 2015), pp. 682–686. ISSN: 0028-0836, 1476-4687. DOI: 10.1038/nature15759. URL: http://www.nature.com/articles/nature15759 (visited on 09/12/2019).

[32] B. Hensen et al. "Loophole-Free Bell Test Using Electron Spins in Diamond: Second Experiment and Additional Analysis". In: *Scientific Reports* 6.1 (Aug. 11, 2016), pp. 1–11. ISSN: 2045-2322. DOI: 10.1038/srep30289. URL: https://www.nature.com/articles/srep30289 (visited on 01/11/2020).

[33] Julian Hofmann et al. "Heralded Entanglement Between Widely Separated Atoms". In: *Science* 337.6090 (July 6, 2012), pp. 72–75. ISSN: 0036-8075, 1095-9203. DOI: 10.1126/science.1221856. pmid: 22767924. URL: https://science.sciencemag.org/content/337/6090/72 (visited on 01/12/2020).

[34] Richard A. Holt. "Atomic Cascade Experiments." 1973. URL: http://adsabs.harvard.edu/abs/1973PhDT........22H (visited on 01/11/2020).

[35] Adrian Kent. "Causal Quantum Theory and the Collapse Locality Loophole". In: *Physical Review A* 72.1 (July 12, 2005), p. 012107. ISSN: 1050-2947, 1094-1622. DOI: 10.1103/PhysRevA.72.012107. URL: https://link.aps.org/doi/10.1103/PhysRevA.72.012107 (visited on 01/11/2020).

[36] Adrian Kent. "Testing Causal Quantum Theory". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 474.2220 (Dec. 2018), p. 20180501. ISSN: 1364-5021, 1471-2946. DOI: 10.1098/rspa.2018.0501. arXiv: 1807.09663. URL: http://arxiv.org/abs/1807.09663 (visited on 01/11/2020).

[37] Carl A. Kocher and Eugene D. Commins. "Polarization Correlation of Photons Emitted in an Atomic Cascade". In: *Physical Review Letters* 18.15 (Apr. 10, 1967), pp. 575–577. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.18.575. URL: https://link.aps.org/doi/10.1103/PhysRevLett.18.575 (visited on 01/25/2020).

[38] Paul G. Kwiat et al. "New High-Intensity Source of Polarization-Entangled Photon Pairs". In: *Physical Review Letters* 75.24 (Dec. 11, 1995), pp. 4337–4341. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.75.4337. URL: https://link.aps.org/doi/10.1103/PhysRevLett.75.4337 (visited on 01/25/2020).

[39] J.Å. Larsson. "Loopholes in Bell Inequality Tests of Local Realism". In: *Journal of Physics A: Mathematical and Theoretical* 47.42 (2014). DOI: 10.1088/1751-8113/47/42/424003.

[40] D. N. Matsukevich et al. "Bell Inequality Violation with Two Remote Atomic Qubits". In: *Physical Review Letters* 100.15 (Apr. 18, 2008), p. 150404. DOI: 10.1103/PhysRevLett.100.150404. URL: https://link.aps.org/doi/10.1103/PhysRevLett.100.150404 (visited on 01/11/2020).

[41] Wayne Myrvold, Marco Genovese, and Abner Shimony. "Bell's Theorem". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Spring 2019. Metaphysics Research Lab, Stanford University, 2019. URL: https://plato.stanford.edu/archives/spr2019/entries/bell-theorem/.

[42] Richard J. Nelson, David G. Cory, and Seth Lloyd. "Experimental Demonstration of Greenberger-Horne-Zeilinger Correlations Using Nuclear Magnetic Resonance". In: *Physical Review A* 61.2 (Jan. 10, 2000), p. 022106. ISSN: 1050-2947, 1094-1622. DOI: 10.1103/PhysRevA.61.022106. URL: https://link.aps.org/doi/10.1103/PhysRevA.61.022106 (visited on 01/24/2020).

[43] Wolfgang Nolting. *Grundkurs Theoretische Physik 5/1*. Springer-Lehrbuch. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013. ISBN: 978-3-642-25402-4 978-3-642-25403-1. DOI: 10.1007/978-3-642-25403-1. URL: http://link.springer.com/10.1007/978-3-642-25403-1 (visited on 12/30/2019).

[44] Dominik Rauch et al. "Cosmic Bell Test Using Random Measurement Settings from High-Redshift Quasars". In: *Physical Review Letters* 121.8 (Aug. 20, 2018), p. 080403. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.121.080403. URL: https://link.aps.org/doi/10.1103/PhysRevLett.121.080403 (visited on 01/11/2020).

[45] M. A. Rowe et al. “Experimental Violation of a Bell’s Inequality with Efficient Detection”. In: *Nature* 409.6822 (Feb. 2001), pp. 791–794. ISSN: 1476-4687. DOI: 10.1038/35057215. URL: https://www.nature.com/articles/35057215 (visited on 01/11/2020).

[46] T. Scheidl et al. “Violation of Local Realism with Freedom of Choice”. In: *Proceedings of the National Academy of Sciences* 107.46 (Nov. 16, 2010), pp. 19708–19713. ISSN: 0027-8424, 1091-6490. DOI: 10.1073/pnas.1002780107. URL: http://www.pnas.org/cgi/doi/10.1073/pnas.1002780107 (visited on 01/11/2020).

[47] E. Schroedinger. “Die gegenwaertige Situation in der Quantenmechanik”. In: *Die Naturwissenschaften* 23.49 (Dec. 1935), pp. 823–828. ISSN: 0028-1042, 1432-1904. DOI: 10.1007/BF01491914. URL: http://link.springer.com/10.1007/BF01491914 (visited on 11/05/2019).

[48] E. Schroedinger. “Die gegenwaertige Situation in der Quantenmechanik”. In: *Die Naturwissenschaften* 23.50 (Dec. 1935), pp. 844–849. ISSN: 0028-1042, 1432-1904. DOI: 10.1007/BF01491987. URL: http://link.springer.com/10.1007/BF01491987 (visited on 11/05/2019).

[49] E. Schroedinger. “Die gegenwärtige Situation in der Quantenmechanik”. In: *Die Naturwissenschaften* 23.48 (Nov. 1935), pp. 807–812. ISSN: 0028-1042, 1432-1904. DOI: 10.1007/BF01491891. URL: http://link.springer.com/10.1007/BF01491891 (visited on 11/05/2019).

[50] F. Selleri. “Einstein Locality and the $$K̂ \bar K̂ $$ System”. In: *Lettere Al Nuovo Cimento Series 2* 36.16 (Apr. 1983), pp. 521–526. ISSN: 1827-613X. DOI: 10.1007/BF02725928. URL: http://link.springer.com/10.1007/BF02725928 (visited on 01/11/2020).

[51] Lynden K. Shalm et al. “Strong Loophole-Free Test of Local Realism”. In: *Physical Review Letters* 115.25 (Dec. 16, 2015), p. 250402. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.115.250402. URL: https://link.aps.org/doi/10.1103/PhysRevLett.115.250402 (visited on 01/11/2020).

[52] Y. H. Shih and C. O. Alley. “New Type of Einstein-Podolsky-Rosen-Bohm Experiment Using Pairs of Light Quanta Produced by Optical Parametric Down Conversion”. In: *Physical Review Letters* 61.26 (Dec. 26, 1988), pp. 2921–2924. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.61.2921. URL: https://link.aps.org/doi/10.1103/PhysRevLett.61.2921 (visited on 01/12/2020).

[53] W. Tittel et al. “Violation of Bell Inequalities by Photons More Than 10 Km Apart”. In: *Physical Review Letters* 81.17 (Oct. 26, 1998), pp. 3563–3566. DOI: 10.1103/PhysRevLett.81.3563. URL: https://link.aps.org/doi/10.1103/PhysRevLett.81.3563 (visited on 01/11/2020).

[54] R. Ursin et al. “Entanglement-Based Quantum Communication over 144 Km”. In: *Nature Physics* 3.7 (July 2007), pp. 481–486. ISSN: 1745-2481. DOI: 10.1038/nphys629. URL: https://www.nature.com/articles/nphys629 (visited on 01/12/2020).

[55] Gregor Weihs et al. “Violation of Bell’s Inequality under Strict Einstein Locality Conditions”. In: *Physical Review Letters* 81.23 (Dec. 7, 1998), pp. 5039–5043. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.81.5039. URL: https://link.aps.org/doi/10.1103/PhysRevLett.81.5039 (visited on 01/11/2020).

[56] Michael M. Wolf and Klaus Hornberger. *Quantum Information Theory*. May 13, 2008. URL: https://www.yumpu.com/de/document/view/21137009/quantum-information-theory-coqus (visited on 08/05/2019).

[57] Juan Yin et al. “Satellite-Based Entanglement Distribution over 1200 Kilometers”. In: *Science* 356.6343 (June 16, 2017), pp. 1140–1144. ISSN: 0036-8075, 1095-9203. DOI: 10.1126/science.aan3211. pmid: 28619937. URL: https://science.sciencemag.org/content/356/6343/1140 (visited on 01/11/2020).

[58] Anton Zeilinger. “Experiment and the Foundations of Quantum Physics”. In: *Reviews of Modern Physics* 71.2 (Mar. 1, 1999), S288–S297. ISSN: 0034-6861, 1539-0756. DOI: 10.1103/RevModPhys.71.S288. URL: https://link.aps.org/doi/10.1103/RevModPhys.71.S288 (visited on 01/11/2020).

[59] M. Żukowski et al. ““Event-Ready-Detectors” Bell Experiment via Entanglement Swapping”. In: *Physical Review Letters* 71.26 (Dec. 27, 1993), pp. 4287–4290. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.71.4287. URL: https://link.aps.org/doi/10.1103/PhysRevLett.71.4287 (visited on 01/21/2020).