Entanglement in many-body systems

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Recent interest in aspects common to quantum information and condensed matter has prompted a flurry of activity at the border of these disciplines that were far distant until a few years ago. Numerous interesting questions have been addressed so far. Here an important part of this field, the properties of the entanglement in many-body systems, are reviewed. The zero and finite temperature properties of entanglement in interacting spin, fermion, and boson model systems are discussed. Both bipartite and multipartite entanglement will be considered. In equilibrium entanglement is shown tightly connected to the characteristics of the phase diagram. The behavior of entanglement can be related, via certain witnesses, to thermodynamic quantities thus offering interesting possibilities for an experimental test. Out of equilibrium entangled states are generated and manipulated by means of many-body Hamiltonians.

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I. INTRODUCTION

Entanglement expresses the "spooky" nonlocality inherent to quantum mechanics (Bell, 1987). Because of that, it gave rise to severe skepticisms since the early days of quantum mechanics. It was only after the seminal contribution of John Bell that the fundamental questions related to the existence of entangled states could be tested experimentally. In fact, under fairly general assumptions, Bell derived a set of inequalities for correlated measurements of two physical observables that any local theory should obey. The overwhelming majority of experiments done so far are in favor of quantum mechanics thus demonstrating that quantum entanglement is physical reality (Peres, 1993).[1]

Entanglement has gained renewed interest with the development of quantum information science (Nielsen and Chuang, 2000). In its framework, quantum entanglement is viewed as a resource in quantum information processing. It is, e.g., believed to be the main ingredient of the quantum speed-up in quantum computation and communication. Moreover, several quantum protocols, such as teleportation (Bennett *et al.*, 1993) just to mention an important example, can be realized exclusively with the help of entangled states.

The role of entanglement as a resource in quantum information has stimulated intensive research trying to unveil both its qualitative and quantitative aspects (Plenio and Vedral, 1998; Wootters, 2001; Bruß, 2002; Vedral, 2002; Bengtsson and Zyczkowski, 2006; Eisert, 2006; Horodecki *et al.*, 2007; Plenio and Virmani, 2007). To this end, necessary criteria for any entanglement measure to be fulfilled have been elaborated and lead to the notion of an entanglement monotone (Vidal, 2000) allowing one to attach a precise number to the entanglement encoded in a given state. There is a substantial bulk of work for bipartite systems, in particular for the case of qubits. Many criteria have been proposed to distinguish separable from entangled pure states, as, for example, the Schmidt rank and the von Neumann entropy. The success in the bipartite case for qubits asked for extensions to the multipartite case, but the situation proved to be far more complicated: Different classes of entanglement occur, which are inequivalent not only under deterministic local operations and classical communication, but even under their stochastic analog (Bennett *et al.*, 2001).

In the last few years it has become evident that quantum information may lead to further insight into other areas of physics as statistical mechanics and quantum field theory (Preskill, 2000). The attention of the quantum information community to systems studied in condensed matter has stimulated an exciting cross-fertilization between the two areas. Methods developed in quantum information have proved to be extremely useful in the analysis of the state of many-body systems. At $T=0$ many-body systems are often described by a complex ground state wave function which contains all correlations that give rise to the various phases of matter (superconductivity, ferromagnetism, quantum Hall systems, etc.). Traditionally many-body systems have been studied by looking, for example, at their response to external perturbations, various order parameters, and excitation spectrum. The study of the ground state of many-body systems with methods developed in quantum information may unveil new properties. At the same time experience built up over the years in condensed matter is helping in finding new protocols for quantum computation and communication: A quantum computer is a many-body system where, different from traditional ones, the Hamiltonian can be controlled and manipulated.

The amount of work at the interface between statistical mechanics and quantum information has grown during the last few years, shining light on many different aspects of both subjects. In particular, there has been extensive analysis of entanglement in quantum critical models (Osborne and Nielsen, 2002; Osterloh *et al.*, 2002; Vidal *et al.*, 2003). Tools from quantum information theory also provided support for numerical methods, such as the density matrix renormalization group or the design of new efficient simulation strategies for many-body systems (see, for example, Verstraete, Porras, and Cirac, 2004; Vidal, 2003, 2004). Spin networks have been proposed as quantum channels (Bose, 2003) by exploiting the collective dynamics of their low lying

[1]There are states that do not violate Bell inequalities and nevertheless are entangled (Methot and Scarani, 2000).

excitations for transporting quantum information.

Despite being at its infancy, this new area of research has grown so fast that a description of the whole field is beyond the scope of a single review. Many interesting facets of this branch of research will therefore remain untouched here. In this review we only discuss the properties of entanglement in many-body systems. The models which will be considered include interacting spin networks, itinerant fermions, harmonic and bosonic systems. All of them are of paramount interest in condensed matter physics.

This review is organized as follows. In the next section we give a brief overview on the concepts and measures of entanglement, with particular attention to those measures that we use later. In Sec. III we then proceed with an introduction to several models of interacting many-body systems which will be the subject of the review. We discuss various aspects of quantum correlations starting from the pairwise entanglement, Sec. IV, we then proceed with the properties of block entropy, Sec. V, and localizable entanglement, Sec. VI. In these three sections it is especially relevant the connection between entanglement and quantum phase transitions. The effect of a finite temperature is considered in Sec. VII. The characterization of entanglement in many-body systems requires also the understanding of multipartite entanglement. This topic will be reviewed in Sec. VIII. From the point of view of quantum information processing, dynamical properties of entanglement are important as well. They will be addressed in Sec. IX. The conclusions, the outlook, and a very short panorama of what is left out from this review are presented in the concluding section.

II. MEASURES OF ENTANGLEMENT

The problem of measuring entanglement is a vast and lively field of research of its own. Numerous different methods have been proposed for its quantification. In this section we do not attempt to give an exhaustive review of the field. Rather we introduce those measures that are largely being used to quantify entanglement in many-body systems. Comprehensive overviews of entanglement measures can be found in Plenio and Vedral (1998); Wootters (2001); Bruß (2002); Vedral (2002); Bengtsson and Zyczkowski (2006); Eisert (2006); Horodecki *et al.* (2007); Plenio and Virmani (2007). In this context, we also outline a method of detecting entanglement, based on entanglement witnesses.

A. Bipartite entanglement in pure states

Bipartite entanglement of pure states is conceptually well understood, although quantifying it for local dimensions higher than two still bears theoretical challenges (Virmani and Plenio, 2000; Horodecki *et al.*, 2007). A pure bipartite state is not entangled if and only if it can be written as a tensor product of pure states of the parts. Moreover, for every pure bipartite state $|\psi_{AB}\rangle$ (with the two parts, A and B), two orthonormal bases $\{|\psi_{A,i}\rangle\}$ and $\{|\phi_{B,j}\rangle\}$ exist such that $|\psi_{AB}\rangle$ can be written as

$$|\psi_{AB}\rangle = \sum_i \alpha_i |\psi_{A,i}\rangle |\phi_{B,i}\rangle, \tag{1}$$

where α_i are positive coefficients. This decomposition is called the Schmidt decomposition and the particular basis coincides with the eigenbasis of the corresponding reduced density operators $\rho_{B/A} = \mathrm{tr}_{A/B}(|\psi_{AB}\rangle) = \Sigma_i \alpha_i^2 |\psi_{B/A,i}\rangle\langle\psi_{B/A,i}|$. The density operators ρ_A and ρ_B have a common spectrum, in particular, they are equally mixed. Since only product states lead to pure reduced density matrices, a measure for their mixedness points a way towards quantifying entanglement in this case. Given the state $|\psi_{AB}\rangle$, we can thus take its Schmidt decomposition, Eq. (1), and use a suitable function α_i to quantify the entanglement.

An entanglement measure E is fixed uniquely after imposing the following conditions: (1) E is invariant under local unitary operations ($\Rightarrow E$ is indeed a function of the α_i's only); (2) E is continuous [in a certain sense also in the asymptotic limit of infinite copies of the state; see, e.g., Plenio and Virmani (2007)]; and (3) E is additive, when several copies of the system are present: $E(|\psi_{AB}\rangle \otimes |\phi_{AB}\rangle) = E(|\psi_{AB}\rangle) + E(|\phi_{AB}\rangle)$. The unique measure of entanglement satisfying the above conditions is the von Neumann entropy of the reduced density matrices:

$$S(\rho_A) = S(\rho_B) = -\sum_i \alpha_i^2 \log(\alpha_i^2), \tag{2}$$

this is just the Shannon entropy of the moduli squared of the Schmidt coefficients. In other words: under the above regularity conditions, the answer to the question of how entangled a bipartite pure state is, given by the von Neumann entropy of (either of) the reduced density matrices. The amount of entanglement is generally difficult to define once we are away from bipartite states, but in several cases we can still gain some insight into many-party entanglement if one considers different bipartitions of a multipartite system.

It is worth noticing that a variety of purity measures are admissible when the third condition on additivity is omitted. In principle, there are infinitely many measures for the mixedness of a density matrix; two of them will typically lead to a different ordering when the Hilbert space of the parts has a dimension larger than 2.

In contrast, if we trace out one of two qubits in a pure state, the corresponding reduced density matrix ρ_A contains only a single independent and unitarily invariant parameter: its eigenvalue $\leqslant 1/2$. This implies that each monotonic function $[0,1/2] \mapsto [0,1]$ of this eigenvalue can be used as an entanglement measure. Though, also here an infinity of different mixedness measures exists, all lead to the same ordering of states with respect to their entanglement, and in this sense all are equivalent. A relevant example is the (one-) tangle (Coffman *et al.*, 2000)

$$\tau_1[\rho_A] = 4 \det \rho_A. \tag{3}$$

By expressing ρ_A in terms of spin expectation values, it follows that $\tau_1[\rho_A]=1-4(\langle S^x\rangle^2+\langle S^y\rangle^2+\langle S^z\rangle^2)$, where $\langle S^\alpha\rangle=\mathrm{tr}_A(\rho_A S^\alpha)$ and $S^\alpha=\frac{1}{2}\sigma^\alpha$, $\sigma^\alpha\{\alpha=x,y,z\}$ being the Pauli matrices. For a pure state $|\psi_{AB}\rangle$ of two qubits the relation $\tau_1 \equiv |\langle\psi^*|\sigma_A^y\otimes\sigma_B^y|\psi\rangle|^2 =: C[|\psi_{AB}\rangle]^2 =: \tau_2$ applies, where C is the concurrence (Hill and Wootters, 1997; Wootters, 1998) for pure states of two qubits, a measure of pairwise entanglement (see the next section), and the asterisk indicates the complex conjugation in the eigenbasis of σ^z. The von Neumann entropy can be expressed as a function of the (one-) tangle $S[\rho_A]=h\{\frac{1}{2}(1+\sqrt{1-\tau_1[\rho_A]})\}$, where $h(x)=:-x\log_2 x-(1-x)\log_2(1-x)$ is the binary entropy.

B. Pairwise qubit entanglement in mixed states

Subsystems of a many-body (pure) state will generally be in a mixed state. In this case a different way of quantifying entanglement can be introduced. Three important representatives are the entanglement cost E_C, the distillable entanglement E_D [both defined by Bennett, Bernstein, *et al.* (1996)], and the entanglement of formation E_F (Bennett, Di Vincenzo, *et al.*, 1996). Whereas E_D and E_C are asymptotic limits of multicopy extraction probabilities of Bell states and creation from such states, the entanglement of formation is the amount of pure state entanglement needed to create a single copy of the mixed state. Although recent progress has been achieved (Paz-Silva and Reina, 2007), the full additivity of the E_F for bipartite systems has not been established yet [see, e.g., Vidal *et al.* (2002)].

The conceptual difficulty behind the calculation of E_F lies in the infinite number of possible decompositions of a density matrix. Therefore even knowing how to quantify bipartite entanglement in pure states, we cannot simply apply this knowledge to mixed states in terms of an average over mixtures of pure state entanglement. The problem is that two decompositions of the same density matrix usually lead to a different average entanglement. Which one do we choose? It turns out that we must take the minimum over all possible decompositions, simply because if there is a decomposition where the average is zero, then this state can be created locally without the need of any entangled pure state, and therefore $E_F=0$. The same conclusion can be drawn from the requirement that entanglement must not increase on average by means of local operations including classical communication (LOCC).

The entanglement of formation of a state ρ is therefore defined as

$$E_F(\rho) := \min \sum_j p_j S(\rho_{A,j}), \tag{4}$$

where the minimum is taken over all realizations of the state $\rho_{AB}=\Sigma_j p_j|\psi_j\rangle\langle\psi_j|$, and $S(\rho_{A,j})$ is the von Neumann entropy of the reduced density matrix $\rho_{A,j}:=\mathrm{tr}_B|\psi_j\rangle\langle\psi_j|$. Equation (4) is the so-called convex roof (also the expression convex hull is found in the literature) of the entanglement of formation for pure states, and a decomposition leading to this convex roof value is called an optimal decomposition.

For systems of two qubits, an analytic expression for E_F is given by

$$E_F(\rho) = -\sum_{\sigma=\pm} \frac{\sqrt{1+\sigma C^2(\rho)}}{2} \ln \frac{\sqrt{1+\sigma C^2(\rho)}}{2}, \tag{5}$$

where $C(\rho)$ is the so-called concurrence (Wootters, 1998, 2001), the convex roof of the pure state concurrence, which has been defined in the previous section. Its convex roof extension is encoded in the positive Hermitian matrix $R\equiv\sqrt{\rho}\tilde{\rho}\sqrt{\rho}=\sqrt{\rho}(\sigma^y\otimes\sigma^y)\rho^*(\sigma^y\otimes\sigma^y)\sqrt{\rho}$, with eigenvalues $\lambda_1^2\geqslant\cdots\geqslant\lambda_4^2$ in the following way:

$$C = \max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\}. \tag{6}$$

As the entanglement of formation is a monotonous function of the concurrence, also C itself or its square τ_2—called also the 2-tangle—can be used as entanglement measures. This is possible due to a curious peculiarity of two-qubit systems: namely, that a continuous variety of optimal decompositions exist (Wootters, 1998). The concurrence C and the tangle τ_1 both range from 0 (no entanglement) to 1.

By virtue of Eq. (6), the concurrence in a spin-1/2 chain can be computed in terms of up to two-point spin correlation functions. As an example we consider a case where the model has a parity symmetry, it is translational invariant, and the Hamiltonian is real; the concurrence in this case reads

$$C_{ij} = 2\max\{0, C_{ij}^{\mathrm{I}}, C_{ij}^{\mathrm{II}}\}, \tag{7}$$

where $C_{ij}^{\mathrm{I}}=|g_{ij}^{xx}+g_{ij}^{yy}|-\sqrt{(1/4+g_{ij}^{zz})^2-M_z^2}$ and $C_{ij}^{\mathrm{II}}=|g_{ij}^{xx}-g_{ij}^{yy}|+g_{ij}^{zz}-1/4$, with $g_{ij}^{\alpha\alpha}=\langle S_i^\alpha S_j^\alpha\rangle$ and $M_z=\langle S^z\rangle$. A state with dominant fidelity of parallel and antiparallel Bell states is characterized by dominant C^{I} and C^{II}, respectively. This was shown by Fubini *et al.* (2006), where the concurrence was expressed in terms of the fully entangled fraction as defined by Bennett, DiVincenzo, *et al.* (1996). Systematic analysis of the relation between the concurrence (together with the 3-tangle, see Sec. II.E) and the correlation functions has been presented by Glaser *et al.* (2003).

The importance of the tangle and concurrence is due to the *monogamy* inequality derived in Coffman *et al.* (2000) for three qubits. This inequality has been proven to hold also for n-qubits system (Osborne and Verstraete, 2006). In the case of many-qubits (the tangle may depend on the site i that is considered) it reads

$$\sum_{j\neq i} C_{ij}^2 \leqslant \tau_{1,i}. \tag{8}$$

The so-called residual tangle $\tau_{1,i}-\Sigma_{j\neq i}C_{ij}^2$ is a measure for multipartite entanglement not stored in pairs of qubits only. We finally mention that the antilinear form of the pure state concurrence was the key for the first explicit construction of a convex roof, and hence its exten-

sion to mixed states (Hill and Wootters, 1997; Wootters, 1998; Uhlmann, 2000).

Another measure of entanglement is the relative entropy of entanglement (Vedral *et al.*, 1997). It can be applied to any number of qubits in principle (or any dimension of the local Hilbert space). It is formally defined as $E(\sigma) := \min_{\rho\in\mathcal{D}} S(\sigma\|\rho)$, where $S(\sigma\|\rho)=\mathrm{tr}\,\sigma[\ln\sigma - \ln\rho]$ is the quantum relative entropy. This relative entropy of entanglement quantifies the entanglement in σ by its distance from the set $\mathcal{D}$ of separable states (since $\mathcal{D}$ is compact, the minimum is assumed always). The main difficulty in computing this measure is to find the disentangled state closest to ρ. This is in general an open problem, even for two qubits. In the presence of certain symmetries—which is the case for, e.g., eigenstates of certain models—an analytical access is possible. In these cases, the relative entropy of entanglement becomes a very useful tool. The relative entropy reduces to the entanglement entropy in the case of pure bipartite states; this also means that its convex roof extension coincides with the entanglement of formation, and is readily deduced from the concurrence (Wootters, 1998).

We close this summary on the pairwise entanglement by commenting on the quantum mutual information. Groisman *et al.* quantified the work necessary to erase the total correlations existing in a bipartite system (Groisman *et al.*, 2005). The entanglement can be erased by a suitable random ensemble of unitary transformations acting on one of the parts, but a certain amount of classical correlation among the two partners may survive. The work necessary to erase all correlations is given by the quantum mutual information

$$\mathcal{I}_{AB} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{9}$$

C. Localizable entanglement

A different approach to entanglement in many-body systems arises from the quest to swap or transmute different types of multipartite entanglement into pairwise entanglement between two parties by means of generalized measures on the rest of the system. In a system of interacting spins on a lattice one could then try to maximize the entanglement between two spins (at positions i and j) by performing measurements on all others. The system is then partitioned in three regions: the sites i, j and the rest of the lattice. This concentrated pairwise entanglement can then be used, e.g., for quantum information processing. A standard example is that the three qubit Greenberger-Horne-Zeilinger (GHZ) state $(1/\sqrt{2})(|000\rangle+|111\rangle)$ after a projective measure in the x direction on one of the sites is transformed into a Bell state.

The concept of localizable entanglement has been introduced by Verstraete (2004); Popp *et al.* (2005). It is defined as the maximal amount of entanglement that can be localized, on average, by doing local measurements in the rest of the system. In the case of N parties, the possible outcomes of the measurements on the remaining $N-2$ particles are pure states $|\psi_s\rangle$ with corresponding probabilities p_s. The localizable entanglement E_{loc} on sites i and j is defined as the maximum of the average entanglement over all possible outcome states $|\psi_s\rangle_{ij}$,

$$E_{\mathrm{loc}}(i,j) = \sup_{\mathcal{E}} \sum_s p_s E(|\psi_s\rangle_{ij}), \tag{10}$$

where $\mathcal{E}$ is the set of all possible outcomes $(p_s,|\psi_s\rangle)$ of measurements and E is the chosen measure of entanglement of a pure state of two qubits (e.g., the concurrence). Although difficult to compute, lower and upper bounds have been found which allow one to deduce a number of consequences for this quantity.

An upper bound to the localizable entanglement is given by the entanglement of assistance (Laustsen *et al.*, 2003) obtained from localizable entanglement when also global and joint measurements were allowed on the $N-2$ spins. A lower bound of the localizable entanglement comes from the following theorem (Verstraete, Martin-Delgado, and Cirac, 2004): Given a (pure or mixed) state of N qubits with reduced correlations $Q_{ij}^{\alpha,\beta}=\langle S_i^\alpha S_j^\beta\rangle - \langle S_i^\alpha\rangle\langle S_j^\beta\rangle$ between the spins i and j and directions α and β then there always exist directions in which one can measure the other spins such that this correlation does not decrease, on average. It then follows that a lower bound to localizable entanglement is fixed by the maximal correlation function between the two parties (one of the various spin-spin correlation functions $Q_{ij}^{\alpha,\beta}$).[2]

D. Entanglement witnesses

It is important to realize that not just the quantification of many-party entanglement is a difficult task; it is an open problem to tell, in general, whether a state of n parties is separable or not. It is therefore of great value to have a tool that is able to merely certify if a certain state is entangled. An entanglement witness W is a Hermitian operator which is able to detect entanglement in a state. The basic idea is that the expectation value of the witness W for the state ρ under consideration exceeds certain bounds only when ρ is entangled. An expectation value of W within this bound, however, does not guarantee that the state is separable. Nonetheless, this is an appealing method also from an experimental point of view, since it is sometimes possible to relate the presence of the entanglement to the measurement of a few observables.

Simple geometric ideas help to explain the witness operator W at work. Let $\mathcal{T}$ be the set of all density matrices and let $\mathcal{E}$ and $\mathcal{D}$ be the subsets of entangled and sepa-

[2]It has been argued (Gour, 2006; Gour and Spekkens, 2006) that in order to extend the entanglement of assistance and localizable entanglement to an entanglement monotone (Vidal, 2000) one should admit also local operations including classical communication on the extracted two spins, this was named entanglement of collaboration.

rable states, respectively. The convexity of $\mathcal{D}$ is a key property for witnessing entanglement. The entanglement witness is then an operator defining a hyperplane which separates a given entangled state from the set of separable states. The main scope of this geometric approach is then to optimize the witness operator (Lewenstein *et al.*, 2000) or to replace the hyperplane by a curved manifold, tangent to the set of separable states (Gühne, 2004) [for other geometric aspects of entanglement see Klyachko (2002), Bengtsson and Zyczkowski (2006), and Leinaas *et al.* (2006)]. We have the freedom to choose W such that $\mathrm{tr}(\rho_D W) \leqslant 0$ for all disentangled states $\rho_D \in \mathcal{D}$. Then, $\mathrm{tr}(\rho W) > 0$ implies that ρ is entangled. A caveat is that the concept of a witness is not invariant under local unitary operations [see, e.g., Cavalcanti and Terra-Cunha (2005)].

Entanglement witnesses are a special case of a more general concept, namely that of positive maps. These are injective superoperators on the subset of positive operators. When we now think of superoperators that act nontrivially only on part of the system (on operators that act nontrivially only on a sub-Hilbert space), then we may ask the question whether a positive map on the subspace is also positive when acting on the whole space. Maps that remain positive also on the extended space are called completely positive maps. The Hermitian time evolution of a density matrix is an example for a completely positive map. Positive but not completely positive maps are important for entanglement theory. There is a remarkable isomorphism between positive maps and Hermitian operators (Jamiolkowski, 1972). This can be used to prove a key theorem (Horodecki *et al.*, 1996): A state ρ_{AB} is entangled if and only if a positive map Λ exists (not completely positive) such that $(1_A \otimes \Lambda_B)\rho_{AB} < 0$. For a two-dimensional local Hilbert space the situation simplifies considerably in that any positive map P can be written as $P = CP_1 + CP_2 T_B$, where CP_1 and CP_2 are completely positive maps and T_B is a transposition operation on subspace B. This decomposition tells us that for a system of two qubits the lack of complete positivity in a positive map is due to a partial transposition. This partial transposition clearly leads to a positive operator if the state is a tensor product of the parts. In fact, also the opposite is true: a state of two qubits ρ_{AB} is separable if and only if $\rho_{AB}^{T_B} \geqslant 0$ that is, its partial transposition is positive. This is very simple to test and it is known as the Peres-Horodecki criterion (Horodecki *et al.*, 1996; Peres, 1996). The properties of entangled states under partial transposition lead to a measure of entanglement known as the negativity. The negativity N_{AB} of a bipartite state is defined as the absolute value of the sum of the negative eigenvalues of $\rho_{AB}^{T_A}$. The logarithmic negativity is then defined as

$$E_N = \log_2 2(2N_{AB} + 1). \tag{11}$$

For bipartite states of two qubits, $\rho_{AB}^{T_A}$ has at most one negative eigenvalue (Sanpera *et al.*, 1998). For general multipartite and higher local dimension this is only a sufficient condition for the presence of entanglement. There exist entangled states with a positive partial transpose known as bound entangled states (Horodecki *et al.*, 1998; Acin *et al.*, 2001).

E. Multipartite entanglement measures

Both the classification of entanglement and its quantification are at a preliminary stage even for distinguishable particles [see, however, Dür *et al.* (2000); Miyake and Wadati (2002); Verstraete *et al.* (2002); Briand *et al.* (2003, 2004), Luque and Thibon (2005); Osterloh and Siewert (2005, 2006); Mandilara *et al.* (2006), and references therein]. We restrict ourselves to those approaches which have been applied so far for the study of condensed matter systems discussed in this review. It has already been mentioned that several quantities are useful as indicators for multipartite entanglement when the whole system is in a pure state. The entropy of entanglement is an example for such a quantity and several works use multipartite measures constructed from and related to it [see, e.g., Coffman *et al.* (2000); Meyer and Wallach (2002); Barnum *et al.* (2003); Scott (2004); de Oliveira, Rigolin, and de Oliveira (2006a); Love *et al.* (2006)]. These measures are of "collective" nature—in contrast to "selective" measures—in the sense that they give indication of a global correlation without discerning among the different entanglement classes encoded in the state of the system.

The geometric measure of entanglement quantifies the entanglement of a pure state through the minimal distance of the state from the set of pure product states (Vedral *et al.*, 1997; Wei and Goldbart, 2003)

$$E_g(\Psi) = -\log_2 \max_{\Phi} |\langle \Psi | \Phi \rangle|^2, \tag{12}$$

where the maximum is on all product states Φ. As discussed in detail by Wei and Goldbart (2003), the previous definition is an entanglement monotone if the convex roof extension to mixed states is taken. It is zero for separable states and rises up to unity for, e.g., the maximally entangled n-particle GHZ states. The difficult task in its evaluation is the maximization over all possible separable states and of course the convex roof extension to mixed states. Despite these complications, a clever use of the symmetries of the problem renders this task accessible by substantially reducing the number of parameters (see Sec. VIII).

Another example for the collective measures of multipartite entanglement as mentioned in the beginning of this section are the measures introduced by Meyer and Wallach (2002) and by Barnum *et al.* (2003, 2004). In the case of qubit system the Q measure of Meyer and Wallach is the average purity [which is the average one-tangle in Coffman *et al.* (2000)] of the state (Meyer and Wallach, 2002; Brennen, 2003; Barnum *et al.*, 2004)

$$E_{\mathrm{gl}} = 2 - \frac{2}{N} \sum_{j=1}^{N} \mathrm{Tr}\, \rho_j^2. \tag{13}$$

The notion of generalized entanglement introduced by Barnum *et al.* (2003, 2004) relaxes the typically chosen partition into local subsystems in real space. The generalized entanglement measure used by Barnum *et al.* is the purity relative to a distinguished Lie algebra $\mathcal{A}$ of observables. For the state $|\psi\rangle$ it is defined as

$$P_{\mathcal{A}} = \mathrm{Tr}\{[\mathcal{P}_{\mathcal{A}}|\psi\rangle\langle\psi|]^2\}, \tag{14}$$

where $\mathcal{P}_{\mathcal{A}}$ is the projection map $\rho \to \mathcal{P}_{\mathcal{A}}(\rho)$. If the set of observables is defined by the operator basis $\{A_1, A_2, \ldots, A_L\}$ then $P_{\mathcal{A}} = \Sigma_{i=1}^{L} \langle A_i \rangle^2$ from which the reduction to Eq. (13) in the case of all local observables is evident. This conceptually corresponds to a redefinition of *locality* as induced by the distinguished observable set, beyond the archetype of partition in the real space. It defines an observer dependent concept of entanglement adapted to, e.g., experimentally accessible or physically relevant observables. In this case, the generalized entanglement coincides with the global entanglement of Meyer and Wallach.

Another approach pursued is the generalization of the concurrence. For the quantification of pairwise entanglement in higher dimensional local Hilbert spaces, the concept of concurrence vectors has been formulated (Audenaert *et al.*, 2001; Badziag *et al.*, 2002) besides the *I*-concurrence (Rungta *et al.*, 2001). A concurrence vector was also proposed for multipartite systems of qubits (Akhtarshenas, 2005). It consists in applying the pure state concurrence formula to a mixed two-site reduced density matrix. It coincides with the true concurrence if and only if the eigenbasis of the density matrices provide optimal decompositions.

The *n-tangle* is a straightforward extension of the concurrence to multipartite states as the overlap of the state with its time reversed (Wong and Christensen, 2001). It vanishes identically for an odd number of qubits, but an entanglement monotone is obtained for an even number of qubits. It detects products of even-site entangled states in addition to certain genuine multipartite entangled states: it detects the multipartite GHZ or cat state, but not, for example, the four qubit cluster state.

Three classes of states inequivalent under SLOCC (stochastic LOCC) exist for four qubits (Osterloh and Siewert, 2005, 2006). Representatives are the GHZ state, the celebrated cluster state, and a third state, which is also measured by the 4-qubit hyperdeterminant. Class selective measures are constructed from two basic elements, namely the operator σ_y employed for the concurrence, and the operator $\sigma_\mu \cdot \sigma^\mu := \mathbb{1}\cdot\mathbb{1} - \sigma^x \cdot \sigma^x - \sigma^z \cdot \sigma^z$ where the dot is a tensor product indicating that the two operators are acting on different copies of the same qubit. Both are invariant under $\mathrm{sl}(2,\mathbb{C})$ operations on the qubit. The 3-tangle is then expressed as $\tau_3[\psi] = \langle\psi*|\cdot\langle\psi*|\sigma_\mu \otimes \sigma_\nu \otimes \sigma_\lambda \cdot \sigma^\mu \otimes \sigma^\nu \otimes \sigma^\lambda|\psi\rangle\cdot|\psi\rangle$. The multilinearity, however, makes it problematic to employ the procedure of convex roof construction presented by Wootters (1998) and Uhlmann (2000) for general mixtures.

Finally we mention the approach pursued by Gühne *et al.* (2005) (see also Sharma and Sharma, 2006) where different bounds on the average energy of a given system were obtained for different types of *n*-particle quantum correlated states. A violation of these bounds then implies the presence of multipartite entanglement in the system. The starting point of Gühne *et al.* is the notion of *n-separability* and *k-producibility* which admit to discriminate particular types of *n*-particle correlations present in the system. A pure state $|\psi\rangle$ of a quantum system of *N* parties is said to be *n*-separable if it is possible to find a partition of the system for which $|\psi\rangle = |\phi_1\rangle|\phi_2\rangle\cdots|\phi_n\rangle$. A pure state $|\psi\rangle$ can be produced by *k*-party entanglement (i.e., it is *k*-producible) if we can write $|\psi\rangle = |\phi_1\rangle|\phi_2\rangle\cdots|\phi_m\rangle$, where $|\phi_i\rangle$ are states of maximally *k* parties; by definition $m \geqslant N/k$. It implies that it is sufficient to generate specific *k*-party entanglement to construct the desired state. Both these indicators for multipartite entanglement are collective, since they are based on the factorizability of a given many particle state into smaller parts. *k*-separability and *k*-producibility both do not distinguish between different *k*-particle entanglement classes [as, e.g., the *k*-particle *W*-states and different *k*-particle graph states (Hein *et al.*, 2004), like the GHZ state].

F. Indistinguishable particles

For indistinguishable particles the wave function is (anti)symmetrized and therefore the definition of entangled states as given in the previous section does not apply. In particular, it does not make sense to consider each individual particle as parts of the partition of the system. Having agreed upon a definition of entanglement, concepts as entanglement cost or distillation remain perfectly valid. Following Ghirardi *et al.* (2002) and Ghirardi and Marinatto (2003) one can address the problem of defining entanglement in an ensemble of indistinguishable particles by seeing if one can attribute to each of the subsystems a complete set of measurable properties, e.g., momenta for free pointless particles. Quantum states satisfying the above requirement represent the separable states for indistinguishable particles.

There is another crucial difference between the entanglement of (indistinguishable) spin-1/2 particles and that of qubits. We therefore consider two fermions on two sites. Whereas the Hilbert space $\mathcal{H}_s$ of a two-site spin lattice has dimension $\dim \mathcal{H}_s = 4$, the Hilbert space $\mathcal{H}_f$ for two fermions on the same lattice has dimension $\dim \mathcal{H}_f = 6$. This is due to the possibility that both fermions, with opposite spins, can be located at the same lattice site. When choosing the following numbering of the states $|1\rangle = f_1^\dagger|0\rangle =: c_{L,\uparrow}^\dagger|0\rangle$, $|2\rangle = f_2^\dagger|0\rangle =: c_{L,\downarrow}^\dagger|0\rangle$, $|3\rangle = f_3^\dagger|0\rangle =: c_{R,\uparrow}^\dagger|0\rangle$, $|4\rangle = f_4^\dagger|0\rangle =: c_{R,\downarrow}^\dagger|0\rangle$, and the definition $|i,j\rangle = f_i^\dagger f_j^\dagger|0\rangle$, there are Bell states analogous to those occurring for distinguishable particles $(|1,3\rangle \pm |2,4\rangle)/\sqrt{2}$ and $(|1,4\rangle \pm |2,3\rangle)/\sqrt{2}$. There are, however, new entangled states, as $(|1,2\rangle \pm |3,4\rangle)/\sqrt{2}$, where both fermions take the same position. The local Hilbert space is made

of four states labeled by the occupation number and the spin, if singly occupied. The site entanglement of indistinguishable particles is then defined as the entanglement of the corresponding Fock states. It can be measured, e.g., by the local von Neumann entropy. This quantity is the analog to the one-tangle for qubits, but the local Hilbert space dimension is four due to the possibility of having empty and doubly occupied sites. Also the quantum mutual information (Groisman *et al.*, 2005), see Eq. (9), can be defined in this way, quantifying the total amount (classical and quantum) of correlations stored in a given state of a second quantized system.

Although from a mathematical point of view the entanglement of indistinguishable particles can be quantified, the major part of the literature on second quantized systems that we discuss in this review considers the site entanglement described above or the entanglement of degrees of freedom, singled out from a suitable set of local quantum numbers (e.g., the spin of the particle at site i). In both cases, entanglement measures for distinguishable particles (see Secs. IV.C.1 and V.F) can be used. With this respect, this section has a different scope than the others on the quantification of entanglement; although most of the discussion which follows will not be used later on, we believe that it will be of interest for further studies of entanglement in itinerant many-body systems.

1. Two fermion entanglement

Due to the antisymmetry under particle exchange, there is no Schmidt decomposition for fermions. Nevertheless, a fermionic analog to the Schmidt rank which classifies entanglement in bipartite systems of distinguishable particles does exist: the so-called Slater rank. A generic state of two electrons on two lattice sites can be written as $|\omega\rangle := \Sigma_{i,j=1}^{4} \omega_{i,j} |i,j\rangle$, where ω is a 4×4 matrix which can be assumed antisymmetric and normalized as $\mathrm{tr}\, \omega^{\dagger}\omega = \frac{1}{2}$. Since the local entities whose entanglement studied here are particles, unitary transformations act on the four-dimensional single-particle Hilbert space. Due to the indistinguishability of particles, the transformation must be the same for each particle. Given a unitary transformation $U \in \mathrm{SU}(4)$ such that $f_j' := U_{jk} f_k$, the transformed state is given by $|\omega'\rangle$, where $\omega' := U\omega U^T$. The above unitary transformation preserves the antisymmetry of ω and can transform every pure state of two spin-1/2 particles on two sites into a state corresponding to the normal form of ω. In fact, every two-particle state within a D-dimensional single-particle Hilbert space can be transformed into the normal form $\omega_s = \mathrm{diag}\{Z_1, \ldots, Z_r, Z_0\}$, where $Z_j = i z_j \sigma_y$ and $(Z_0)_{ij} = 0$ for $i,j \in \{1, \ldots, D-2r\}$. In the previous expression r is then called the Slater rank of the pure fermion state (Schliemann, Cirac, *et al.*, 2001; Schliemann, Loss, and MacDonald, 2001; Eckert *et al.*, 2002). A pure fermion state is entangled if and only if its Slater rank is larger than 1. It is important to note that the above concept of entanglement only depends on the dimension of the Hilbert space accessible to each of the particles (this includes indistinguishable particles on a single D-level system).

For electrons on an L-site lattice the "local" Hilbert space dimension is $2L$, and the question whether a pure state living in a $2L$-dimensional single particle Hilbert space has full Slater rank can be answered by considering the Pfaffian of ω (Caianello and Fubini, 1952; Muir, 1960):

$$\sum_{\pi \in \mathcal{S}_{2L}^{<}} \mathrm{sgn}(\pi) \prod_{j=1}^{L} \omega_{\pi(2j-1),\pi(2j)}, \tag{15}$$

which is nonzero only if ω has full Slater rank L. In the above definition $\mathcal{S}_{2L}^{<}$ denotes those elements π of the symmetric group $\mathcal{S}_{2L}$ with ordered pairs, i.e., $\pi(2m-1) < \pi(2m)$ for all $m \leqslant L$ and $\pi(2k-1) < \pi(2m-1)$ for $k < m$. Note that relaxing the restriction to $\mathcal{S}_{2L}^{<}$ leads to a combinatorial factor of $2^L L!$ by virtue of the antisymmetry of ω and hence can we write

$$\mathrm{pf}[\omega] = \frac{1}{2^L L!} \sum_{j_1, \ldots j_{2L}=1}^{2L} \varepsilon^{j_1, \ldots j_{2L}} \omega_{j_1 j_2} \ldots \omega_{j_{2L-1} j_{2L}}, \tag{16}$$

where $\varepsilon^{j_1, \ldots j_{2L}}$ is the fully antisymmetric tensor with $\varepsilon^{1,2,\ldots,2L} = 1$. There is a simple relation between the Pfaffian and the determinant of an antisymmetric even-dimensional matrix: $\mathrm{pf}[\omega]^2 = \det[\omega]$.

For the simplest case of two spin-1/2 fermions on two lattice sites the Pfaffian reads $\mathrm{pf}[\omega] = \omega_{1,2}\omega_{3,4} - \omega_{1,3}\omega_{2,4} + \omega_{1,4}\omega_{2,3}$. Normalized in order to range in the interval [0, 1] this has been called the fermionic concurrence $\mathcal{C}[|\omega\rangle]$ (Schliemann, Cirac, *et al.*, 2001; Schliemann, Loss, and MacDonald, 2001; Eckert *et al.*, 2002):

$$\mathcal{C}[|\omega\rangle] = |\langle \tilde{\omega} | \omega \rangle| = 8 |\mathrm{pf}[\omega]|, \tag{17}$$

where $\tilde{\omega} := \frac{1}{2} \varepsilon^{ijkl} \omega_{k,l}^{*}$ has been termed the *dual* to ω. Then, $|\tilde{\omega}\rangle =: \mathcal{D}|\omega\rangle$ is the analog to the conjugated state in (Hill and Wootters, 1997; Wootters, 1998; Uhlmann, 2000) leading to the concurrence for qubits. It is important to note that the Pfaffian in Eq. (15) is invariant under the complexification of su($2L$), since it is the expectation value of an antilinear operator, namely the conjugation $\mathcal{D}$ for state $|\omega\rangle$. Since this invariant is a bilinear expression in the state coefficients, its convex roof is readily obtained (Uhlmann, 2000) by means of the positive eigenvalues λ_i^2 of the 6×6 matrix $R = \sqrt{\rho}\mathcal{D}\rho\mathcal{D}\sqrt{\rho}$. The conjugation $\mathcal{D}$ expressed in the basis $\{|1,2\rangle, |1,3\rangle, |1,4\rangle, |2,3\rangle, |2,4\rangle, |3,4\rangle\}$ takes the form $\mathcal{D}_0\mathcal{C}$, where $\mathcal{C}$ is the complex conjugation and the only nonzero elements of $\mathcal{D}$ are $\mathcal{D}_{16} = \mathcal{D}_{61} = \mathcal{D}_{34} = \mathcal{D}_{43} = 1$ and $\mathcal{D}_{25} = \mathcal{D}_{52} = 1$. Notice that the center part of this matrix is precisely $\sigma_y \otimes \sigma_y$ and indeed corresponds to the Hilbert space of two qubits. The remaining part of the Hilbert space gives rise to an entanglement of different values for the occupation number. This type of entanglement has been referred to as the "fluffy bunny" (Verstraete and Cirac, 2003; Wiseman *et al.*, 2003), in the literature.

For a single-particle Hilbert space with dimension larger than four one encounters similar complications as

for two distinguishable particles on a bipartite lattice and local Hilbert space dimension larger than two, i.e., for two *qudits*. This is because different classes of entanglement occur, which are characterized by different Slater rank as opposed to their classification by different Schmidt rank for distinguishable particles. The Slater rank can be obtained by looking at Pfaffian minors (Muir, 1960): if the Slater rank is r, all Pfaffian minors of dimension larger than $2r$ are identically zero.

2. Multipartite entanglement for fermions

For indistinguishable particles the only classification available up to now is to check whether or not a pure state has Slater rank one. Eckert *et al.* formulated two recursive lemmata (Eckert *et al.*, 2002) that can be summarized as follows: Let an N-electron state be contracted with $N-2$ arbitrary single electron states encoded in the vectors $\mathbf{a}^j$ as $\mathbf{a}_k^j f_k^\dagger|0\rangle$ ($j=1,\ldots,N-2$ and sum convention) to a two-electron state. Then the Pfaffian of the two-electron state is zero if and only if the original state (and hence all intermediate states in a successive contraction) has Slater rank one. This means that all four-dimensional Pfaffian minors of ω are zero.

Instead of the Pfaffian of ω, the single-particle reduced density matrix can also be considered, and its von Neumann entropy as a measure for the quantum entanglement has been analyzed by Li *et al.* (2001) and Paškauskas and You (2001). It is important to remember that for distinguishable particles the local reduced density matrix has rank one if and only if the original state were a product. This is no longer true for indistinguishable particles. For an N-particle pure state with Slater rank one the rank of the single-particle reduced density matrix coincides with the number of particles N. A subtlety is that a measure of entanglement is obtained after subtraction of the constant value of the von Neumann entropy of a disentangled state. This must be taken into account also for the extension of the measure to mixed states.

3. Entanglement of particles

Entanglement in the presence of superselection rules (SSR) induced by particle conservation has been discussed by Bartlett and Wiseman (2003), Schuch *et al.* (2003, 2004), and Wiseman and Vaccaro (2003). The main difference in the concept of entanglement of particles (Wiseman and Vaccaro, 2003) from the entanglement of indistinguishable particles as described in the preceding section (but also to that obtained from the reduced density matrix of, e.g., spin degrees of freedom of indistinguishable particles) consists in the projection of the Hilbert space onto a subspace of fixed particle numbers for either part of a bipartition of the system. The bipartition is typically chosen to be spacelike, as motivated from experimentalists or detectors sitting at distinct positions. For example, two experimentalists in order to detect the entanglement between two indistinguishable particles must have one particle each in their laboratory.

This difference induced by particle number superselection is very subtle and shows up if multiple occupancies occur at single sites for fermions with some inner degrees of freedom, such as spin. Their contribution is finite for finite discrete lattices and will generally scale to zero in the thermodynamic limit with vanishing lattice spacing. Therefore both concepts of spin entanglement of two distant particles coincide in this limit. Significant differences are to be expected only for finite nondilute systems. It must be noted that the same restrictions imposed by SSR which change considerably the concept of entanglement quantitatively and qualitatively, on the other hand, enable otherwise impossible protocols of quantum information processing (Schuch *et al.*, 2003, 2004) which are based on variances about the observable fixed by superselection.

Wiseman and Vaccaro projected an N-particle state $|\psi_N\rangle$ onto all possible subspaces, where the two parties have a well defined number $(n_A, n_B=N-n_A)$ of particles (Wiseman and Vaccaro, 2003). Let $|\psi[n_A]\rangle$ be the respective projection, and let p_{n_A} be the weight $\langle\psi[n_A]|\psi[n_A]\rangle/\langle\psi_N|\psi_N\rangle$ of this projection. Then the entanglement of particles E_p is defined as

$$E_p[|\psi_n\rangle] = \sum_n p_n E_M[\psi[n_A]], \tag{18}$$

where E_M is some measure of entanglement for distinguishable particles. Although this certainly represents a definition of entanglement appealing for experimental issues, it is sensitive only to situations where, e.g., the two initially indistinguishable particles eventually are separated and can be examined one by one by Alice and Bob. Consequently, "local operations" have been defined by Wiseman and Vaccaro (2003) as those performed by Alice and Bob in their laboratory after having measured the number of particles.[3]

Verstraete and Cirac pointed out that the presence of SSR gives rise to a new resource which has to be

[3]As a potential difference between the entanglement of photons as opposed to that of massive bosonic particles, it has been claimed that certain superselection rules may hold for massive particles only. One such claim is that we would in practice not be able to build coherent superpositions of states containing a different number of massive particles [for a recent discussion see Bartlett *et al.* (2007)]. This superselection rule would, for instance, prohibit creating a superposition of a hydrogen atom and molecule. However, the origin and validity of any superselection rule remains a debated subject. The arguments pro superselection rules usually involve some symmetry considerations, or some decoherence mechanism. On the other hand, it turns out that if we allow most general operations in quantum mechanics, we no longer encounter any superselection restrictions. Recent work (Dowling, Bartlett, *et al.*, 2006; Terra Cunha *et al.*, 2007) has shown that it should be possible to coherently superpose massive particles and to observe a violation of certain Bell inequalities (Terra Cunha *et al.*, 2007) also for this case.

quantified. They have proposed to replace the quantity E_p with the SSR entanglement of formation. This is defined as

$$E_f^{(\mathrm{SSR})}[|\psi_N\rangle] = \min_{p_n,\psi_n} \sum_n p_n E_M[\psi_n],$$

where the minimization is performed over all those decomposition of the density matrix where the $|\psi\rangle_n$ are eigenstates of the total number of particles (Schuch *et al.*, 2003, 2004).

4. Entanglement for bosons

The quantification and classification of boson entanglement is very close in spirit to that of fermions as described in Sec. II.F.1. In the bosonic case the matrix ω introduced in the previous section is symmetric under permutations of the particle numbers. Consequently, for any two-particle state of indistinguishable bosons, ω can be diagonalized by means of unitary transformations of the single-particle basis. This leads to the Schmidt decomposition for bosons (Eckert *et al.*, 2002). A curious feature distinguishing this case from the entanglement measures of distinguishable particles is that the Schmidt decomposition is not unique. In fact, any two equal Schmidt coefficients admit for a unitary transformation of the two corresponding basis states, such that the superposition of the two doubly occupied states can be written as a symmetrized state of two orthogonal states (Li *et al.*, 2001; Ghirardi and Marinatto, 2005). This is the reason why it is not directly the Schmidt rank, but rather the reduced Schmidt rank—obtained after having removed all double degeneracies of the Schmidt decomposition—that determines whether or not a state is entangled. This nonuniqueness of the Schmidt rank is also responsible for the ambiguity of the von Neumann entropy or other purity measures of the single-particle reduced density matrix as an entanglement measure for bosons (Ghirardi and Marinatto, 2005).

With z_i the Schmidt coefficients with degeneracy g_i, the reduced Schmidt rank is at most $g_i/2+2\{g_i/2\}$, where $\{\cdot\}$ denotes the noninteger part. As a consequence, a Schmidt rank larger than 2 implies the presence of entanglement. A Schmidt rank 2 with degenerate Schmidt coefficients can be written as a symmetrized product of orthogonal states and consequently is disentangled (Ghirardi and Marinatto, 2005). This feature is also present in the N-boson case, where in the presence of up to N-fold degenerate Schmidt coefficients the corresponding state can be rewritten as a symmetrization of a product.

For bipartite systems ω has full Schmidt rank if $\det \omega \neq 0$. A Schmidt rank 1 can be verified by the same contraction technique described for the fermion case in the previous section, where the Pfaffian must be replaced by the determinant. This applies to both the bipartite and the multipartite case (Eckert *et al.*, 2002).

G. Entanglement in harmonic systems

In this section we concentrate on the entanglement between distinct modes of harmonic oscillators [see Braunstein and van Loock (2005); Adesso and Illuminati (2007) for recent reviews on the subject]. The entanglement in this case is termed as continuous variable entanglement in the literature (to be distinguished from the entanglement of indistinguishable bosonic particles; see Sec. II.F).

Dealing with higher dimensional space of the local degrees of freedom generally involves complications which are not tamable within the current knowledge about entanglement. The Peres-Horodecki criterion, just to mention an important example, is not sufficient already for two three-level systems, 3×3. The situation simplifies if only so-called Gaussian states of the harmonic oscillator modes are considered. This restriction makes the infinite dimensional case even conceptually simpler than the finite dimensional counterparts. In order to explain what Gaussian states are, we introduce the Wigner distribution function $W(p,q)$ (Wigner, 1932). For a single degree of freedom it is defined from the density operator ρ as

$$W(r,p) := \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dr' \langle r+r'|\rho|r-r'\rangle e^{(2i/\hbar)pr'}, \tag{19}$$

where r and p are conjugate position and momentum variables of the degree of freedom. The connection between bosonic operators $\hat{a}$, $\hat{a}^\dagger$ and phase space operators $\hat{r}$, $\hat{p}$ is $\hat{a}=(\hat{r}+i\hat{p})/\sqrt{2}$, $\hat{a}^\dagger=(\hat{a})^\dagger=(\hat{r}-i\hat{p})/\sqrt{2}$. More degrees of freedom are taken into account in a straightforward manner. A (mixed) state ρ is then called Gaussian when its Wigner distribution function is Gaussian. Examples for such states are coherent pure states $|\alpha\rangle$, $\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$ with $\alpha\in\mathbb{C}$, and arbitrary mixtures of coherent states $\rho=\int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$, determined by the so-called P-distribution $P(z)$. Such states are termed classical if the Wigner function and the P-distribution are nonnegative [see Simon (2000)].

The key quality of Gaussian states is that they are completely classified by second moments, which are encoded in the symmetric so-called (co-)variance matrix with the uncertainties of the phase space coordinates as entries. For two bosonic modes the phase space is four dimensional and the covariance matrix V is defined as

$$V_{\alpha\beta} := \langle\{\Delta\hat{\xi}_\alpha,\Delta\hat{\xi}_\beta\}\rangle = \int d^4\xi \Delta\xi_\alpha\Delta\xi_\beta W(\{\xi_\gamma\}), \tag{20}$$

where the curly brackets on the left-hand side indicate the anticommutator. The components of $\hat{\xi}_\alpha$, $\alpha=1,\ldots,4$ are $(\hat{r}_1,\hat{p}_1,\hat{r}_2,\hat{p}_2)$ and $\Delta\hat{\xi}_\alpha := \hat{\xi}_\alpha - \langle\hat{\xi}_\alpha\rangle$; the average $\langle\cdot\rangle$ is taken with respect to the given two-mode density matrix ρ, or, equivalently, using the Wigner distribution of ρ. Then, the canonical commutation relations assume the compact form $[\hat{\xi}_\alpha,\hat{\xi}_\beta]=i\hbar\Omega_{\alpha,\beta}$ with $\Omega=i\sigma_y\otimes\mathbb{1}$. When expressed in terms of V, the Heisenberg uncertainty relation can be invoked in invariant form with respect to

canonical transformations as $\det V \geq \hbar^2/4$ [see, e.g., Simon *et al.* (1994)]. The set of the real linear canonical transformation generates the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ that plays an important role in the theory. Being a symplectic matrix, V can be brought in its diagonal form V_n by means of symplectic transformations. The elements on the diagonal are then called the symplectic eigenvalues of V. An analysis of $V_n\Omega$ has unveiled an even more powerful invariant form of the Heisenberg uncertainty principle, $V+(i\hbar/2)\Omega \geq 0$, where the positive semidefiniteness means that all symplectic eigenvalues are nonnegative. The uncertainty relation can hence be cast directly in terms of the symplectic eigenvalues of the covariance matrix V, which are the absolute values of the eigenvalues of $-i\Omega V$.

Some aspects of the harmonic systems can be disclosed by recognizing that the Gaussian structure of the bosonic states can be thought of as a certain limit of the algebraic structure of the qubits in the sense that $\mathrm{Sp}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{C})$. The latter is the invariance group relevant for qubit entanglement classification and quantification (Dür *et al.*, 2000; Verstraete *et al.*, 2003; Osterloh and Siewert, 2005).

We now introduce the notion of bipartite entanglement for Gaussian states. In complete analogy to the finite-dimensional case, a state is termed separable if it is a mixture of product states. In particular, all classical states, i.e.,

$$\rho = \int d^2z_1 d^2z_2 P(z_1,z_2) |z_1\rangle\langle z_1| \otimes |z_2\rangle\langle z_2|, \tag{21}$$

with positive $P(z_1,z_2)$ are separable.

It was Simon (2000) that first proved the Peres-Horodecki positive partial transpose criterion being necessary and sufficient for entanglement of two harmonic oscillator modes, again in complete analogy to a system of two qubits. The effect of transposition of the density matrix is a sign change in the momentum variables of the Wigner function (19). Consequently, a partial transposition induces a sign change of those momenta in the phase space vector, where the transposition should act on. For an entangled state, the partial transposition $\tilde{V}$ of its covariance matrix V might then have symplectic eigenvalues smaller than $\hbar/2$. This can be detected by the logarithmic negativity as defined from the symplectic (doubly degenerate) eigenvalues $\{\tilde{c}_i; i=1,\ldots,n\}$ of $\tilde{V}/\hbar$ (Vidal and Werner, 2002):

$$E_N(V) = -\sum_{i=1}^{n} \log_2(2\tilde{c}_i). \tag{22}$$

These important results paved the way towards a systematic analysis of multipartite systems of distinguishable bosonic modes.

III. MODEL SYSTEMS

This section is devoted to the basic properties of the model systems that will be analyzed in the rest of the review (in several cases we concentrate on one-dimensional systems).

A. Spin models

Interacting spin models (Auerbach, 1998; Schollwöck *et al.*, 2004) provide a paradigm to describe a wide range of many-body systems. They account for the effective interactions in a variety of different physical contexts ranging from high energy to nuclear physics (Polyakov, 1977; Belitsky *et al.*, 2004). In condensed matter beside describing the properties of magnetic compounds [see Matsumoto *et al.* (2004) for a recent survey], they capture several aspects of high-temperature superconductors, quantum Hall systems, and heavy fermions, just to mention few important examples. Hamiltonians for interacting spins can be realized artificially in Josephson junctions arrays (Fazio and van der Zant, 2001) or with neutral atoms loaded in optical lattices (Duan *et al.*, 2003; Jané *et al.*, 2003; Porras and Cirac, 2004). Interacting spins are paradigm systems for quantum information processing (Nielsen and Chuang, 2000).

1. Spin-1/2 models with short range interactions

A model Hamiltonian for a set of localized spins interacting with nearest-neighbor exchange coupling on a d-dimensional lattice is

$$\mathcal{H}(\gamma,\Delta,h_z/J) = \frac{J}{2}\sum_{\langle i,j\rangle}[(1+\gamma)S_i^xS_j^x + (1-\gamma)S_i^yS_j^y] + J\Delta\sum_{\langle i,j\rangle} S_i^zS_j^z - h_z\sum_i S_i^z. \tag{23}$$

In the previous expression i,j are lattice points, $\langle\cdot\rangle$ constraints the sum over nearest neighbors, and S_i^α ($\alpha = x,y,z$) are spin-1/2 operators. A positive (negative) exchange coupling J favors antiferromagnetic (ferromagnetic) ordering in the x-y plane. The parameters γ and Δ account for the anisotropy in the exchange coupling in the z direction, and h_z is the transverse magnetic field. There are only very few exact results concerning $\mathcal{H}(\gamma,\Delta,h_z/J)$ in dimension $d>1$. The ground state of Eq. (23) is in general entangled. It exists, however, for any value of the coupling constants γ and Δ, $J>0$ a point in $d=1,2$ (for bipartite lattices) where the ground state is factorized (Kurmann *et al.*, 1982; Roscilde *et al.*, 2005b). It occurs at the so-called factorizing field h_f given by $h_f=(z/2)J\sqrt{(1+\Delta)^2-(\gamma/2)^2}$, where z is the coordination number.

In $d=1$ the model is exactly solvable in several important cases. In the following we discuss the anisotropic quantum XY model ($\Delta=0$ and $0\leq\gamma\leq1$) and the XXZ model ($\gamma=0$). Also the XYZ model in zero field, $\gamma\neq0, \Delta\neq0$ can be solved exactly but it will not be discussed here [see Takahashi (1999) for a review].

(*a*) $\Delta=0$: *Quantum XY model*. The quantum Ising model corresponds to $\gamma=1$ while the (isotropic) XX model is obtained for $\gamma=0$. In the isotropic case the

model possesses an additional symmetry resulting in the conservation of magnetization along the z axis. For any value of the anisotropy the model can be solved exactly (Lieb *et al.*, 1961; Pfeuty, 1970; Barouch and McCoy, 1971). By first applying the Jordan-Wigner transformation $c_k = e^{i\pi\sum_{j=1}^{k-1}\sigma_j^+\sigma_j^-}\sigma_k^-$ [with $\sigma^\pm = (1/2)(\sigma^x \pm i\sigma^y)$] the XY model can be transformed onto a free fermion Hamiltonian

$$H = \sum_{i,j}\left[c_i^\dagger A_{i,j}c_j + \frac{1}{2}(c_i^\dagger B_{i,j}c_j^\dagger + \text{H.c.})\right] + \frac{1}{2}\sum_i A_{i,i}. \quad (24)$$

In the previous equation $c_i, c_i^\dagger$ are the annihilation and creation operators for the spinless Jordan-Wigner fermions. The two matrices $\mathbf{A}$, $\mathbf{B}$ are defined as $A_{j,k} = -J(\delta_{k,j+1} + \delta_{j,k+1}) - h_z\delta_{j,k}$ and $B_{j,k} = -\gamma J(\delta_{k,j+1} - \delta_{j,k+1})$. For the case of periodic boundary conditions on the spins, an extra boundary term appears in the fermionic Hamiltonian which depends on the parity of the total number of fermions N_F. Note that although N_F does not commute with the Hamiltonian the parity of N_F is conserved. A generic quadratic form, like Eq. (24), can be diagonalized in terms of the normal-mode spinless Fermi operators by first going to the Fourier space and then performing a Bogoliubov transformation.

The properties of the Hamiltonian are governed by the dimensionless coupling constant $\lambda = J/2h$. In the interval $0 < \gamma \leqslant 1$ the system undergoes a second order quantum phase transition at the critical value $\lambda_c = 1$. The order parameter is magnetization in the x direction, $\langle S^x \rangle$, which is different from zero for $\lambda > 1$. In the ordered phase the ground state has a twofold degeneracy reflecting a global phase flip symmetry of the system. Magnetization along the z direction, $\langle S^z \rangle$ is different from zero for any value of λ, but presents a singular behavior of its first derivative at the transition. In the interval $0 < \gamma \leqslant 1$ the transition belongs to the Ising universality class. For $\gamma = 0$ the quantum phase transition is of the Berezinskii-Kosterlitz-Thouless type.

As discussed in Secs. II.A and II.B one- and two-site-entanglement measures can be related to various equal-time spin correlation functions (in some cases the block entropy can be reduced to the evaluation of two-point correlators) $M_l^\alpha(t) = \langle\psi|S_l^\alpha(t)|\psi\rangle$ and $g_{lm}^{\alpha\beta}(t) = \langle\psi|S_l^\alpha(t)S_m^\beta(t)|\psi\rangle$. These correlators have been calculated for this class of models in the case of thermal equilibrium (Lieb *et al.*, 1961; Pfeuty, 1970; Barouch and McCoy, 1971). These can be recast in the form of Pfaffians that for stationary states reduce to Toeplitz determinants (i.e., determinants, whose entries depend only on the difference of their row and column number). It can be demonstrated that the equal-time correlation functions can be expressed as a sum of Pfaffians (Amico and Osterloh, 2004).

(*b*) $\gamma = 0$: *XXZ model*. The two isotropic points $\Delta = 1$ and $\Delta = -1$ describe the antiferromagnetic and ferromagnetic chains, respectively. In one dimension the XXZ Heisenberg model can be solved exactly by the Bethe ansatz technique (Bethe, 1931) [see, e.g., Takahashi (1999)] and the correlation functions can be expressed in terms of certain determinants [see Bogoliubov *et al.* (1993) for a review]. Correlation functions, especially for intermediate distances, are in general difficult to evaluate, although important steps in this direction have been made (Kitanine *et al.*, 1999; Göhmann and Korepin, 2000).

The zero temperature phase diagram of the XXZ model in zero magnetic field shows a gapless phase in the interval $-1 \leqslant \Delta < 1$. Outside this interval the excitations are gapped. The two phases are separated by a Berezinskii-Kosterlitz-Thouless phase transition at $\Delta = 1$ while at $\Delta = -1$ the transition is first order. In the presence of the external magnetic field a finite energy gap appears in the spectrum. The universality class of the transition is not affected as a result of the conservation of the total spin-z component (Takahashi, 1999).

When one moves away from one dimension, exact results are rare. Nevertheless it is now established that the ground state of a two-dimensional antiferromagnet possesses Néel long range order (Manousakis, 1991; Dagotto, 1994).

2. Spin-1/2 models with infinite range interaction

In this case each spin interacts with all other spins in the system with the same coupling strength $\mathcal{H} = -(J/2)\Sigma_{ij}[S_i^x S_j^x + \gamma S_i^y S_j^y] - \Sigma_i \mathbf{h}_i \cdot \mathbf{S}_i$. For site-independent magnetic field $h_i^\alpha = h_\alpha \forall i$, this model was originally proposed by Lipkin, Meshkov, and Glick (LMG) (Lipkin *et al.*, 1965; Meshkov, Glick, and Lipkin, 1965; Meshkov, Lipkin, and Glick, 1965) to describe a collective motion in nuclei. In this case the dynamics of the system can be described in terms of a collective spin $S_\alpha = \Sigma_j S_j^\alpha$. The previous Hamiltonian reduces to

$$\mathcal{H} = -\frac{J}{2}[(S^x)^2 + \gamma(S^y)^2] - \mathbf{h}\cdot\mathbf{S}. \quad (25)$$

Since the Hamiltonian commutes with the Casimir operator $\mathbf{S}^2$ the eigenstates can be labeled by the representation S of the collective spin algebra, at most linear in the number N of spins; this reduces (from 2^N to $N/2$) the complexity of the problem. A further simplification is achieved at the supersymmetric point corresponding to $J^2\gamma = 4h_z$, where the Hamiltonian can be factorized in two terms linear in the collective spin (Unanyan and Fleischhauer, 2003); then the ground state can be obtained explicitly. For a ferromagnetic coupling ($J > 0$) and $h_x = h_y = 0$ the system undergoes a second order quantum phase transition at $\lambda_c = 1$, characterized by mean field critical indices (Bottet *et al.*, 1982). The average magnetization (for any γ) $m_z = \langle S_z \rangle / N$ saturates for $\lambda \leqslant \lambda_c$ while it is suppressed for $\lambda > \lambda_c$. For $h_y = 0$, $h_z < 1$, and $\gamma = 0$ the model exhibits a first order transition at $h_x = 0$ (Vidal *et al.*, 2006) while for an antiferromagnetic coupling and $h_y = 0$ a first order phase transition at $h_z = 0$ occurs, where the magnetization saturates abruptly at the same value $m^z = 1/2$ for any γ's.

The model Hamiltonian introduced at the beginning of this section embraces an important class of interacting fermion systems as well. By interpreting the nonhomogeneous magnetic field as a set of energy levels $(h_z)_i \equiv -\epsilon_i$, for $h_x=h_y=0$ and $\gamma=1$, it expresses the BCS model. This can be realized by noting that the operators $S_j^- := c_{j\uparrow}c_{j\downarrow}, S_j^+ := (S_j^-)^\dagger$, $S_j^z := (c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j\downarrow}^\dagger c_{j\downarrow} - 1)/2$ span the su(2) algebra in the representation 1/2. In the fermion language the Hamiltonian reads $\mathcal{H}_{\mathrm{BCS}} = \Sigma_{j,\sigma=\{\uparrow,\downarrow\}} \epsilon_j c_{j\sigma}^\dagger c_{j\sigma} - (J/2)\Sigma_{ij} c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger c_{i\downarrow} c_{i\uparrow}$.

Both the LMG and BCS type models can be solved exactly by Bethe ansatz (Richardson, 1963; Richardson and Sherman, 1964) [see also Dukelsky *et al.* (2004) for a review] as they are quasiclassical descendants of the six vertex model (Amico *et al.*, 2001; Di Lorenzo *et al.*, 2002; Ortiz *et al.*, 2005).

3. Frustrated spin-1/2 models

Frustration arises in systems where certain local constraints prevent the system from reaching a local energy minimum. The constraints can be of geometric nature (for example, the topology of the underlying lattice) or of dynamical nature (two terms in the Hamiltonian tending to favor incompatible configurations). A classical example of the first type is that of an antiferromagnet in a triangular lattice with Ising interaction. At a quantum mechanical level this phenomenon can result in the appearance of ground state degeneracies. The equilibrium and dynamical properties of frustrated systems have been extensively studied in the literature (Diep, 2005) in both classical and quantum systems.

A prototype of frustrated models in one dimension is the antiferromagnetic Heisenberg model with nearest- and next-nearest-neighbor interactions. This class of models was discussed originally to study the spin-Peierls transition (Schollwöck *et al.*, 2004). The Hamiltonian reads

$$H_\alpha = J\sum_{i=1}^{N} (\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \alpha \mathbf{S}_i \cdot \mathbf{S}_{i+2}). \tag{26}$$

Analytical calculations (Haldane, 1982) corroborated by numerical result (Okamoto and Nomura, 1992) indicate that at $\alpha \approx 1/4$ there is a quantum phase transition to a dimerized two-fold degenerate ground state, where singlets are arranged on doubled lattice constant distances. Such a phase is characterized by a finite gap in the low lying excitation spectrum.

The Majumdar-Ghosh model (Majumdar and Ghosh, 1969a, 1969b; Majumdar, 1970) is obtained from Eq. (26) for $\alpha=1/2$. The exact ground state can be solved by means of matrix product states (see next section) and it is shown to be disordered. It is a doubly degenerate valence bond state made of nearest-neighbor spin singlets. Although all two-point correlation functions vanish, a finite four-spin correlation function does reflect an ordered dimerization.

FIG. 1. Schematic of the nearest-neighbor valence bond state, exact ground state of the spin-1 model in Eq. (27) for $\beta=1/3$ (AKLT-model). The ground state is constructed regarding every $S=1$ in the lattice sites as made of a pair of $S=1/2$, and projecting out the singlet state. The singlets are then formed taking pairs of $S=1/2$ in nearest-neighbor sites.

4. Spin-1 models

Spin-1 systems were originally considered to study the quantum dynamics of magnetic solitons in antiferromagnets with single ion anisotropy (Mikeska, 1995). In one dimension, half-integer and integer spin chains have very different properties (Haldane, 1983a 1983b). Long range order which is established in the ground state of systems with half-integer spin (Lieb *et al.*, 1961) may be washed out for integer spins. In this latter case, the system has a gap in the excitation spectrum. A paradigm model of interacting spin-1 systems is

$$H = \sum_{i=0}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \beta(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2. \tag{27}$$

The resulting gapped phase arises because of the presence of zero as an eigenvalue of S_i^z; the corresponding eigenstates represent a spin excitation that can move freely in the chain, ultimately disordering the ground state of the system (Gomez-Santos, 1991; Mikeska, 1995). The so-called string order parameter was proposed to capture the resulting "floating" Néel order, made of alternating spins $|\uparrow\rangle$, $|\downarrow\rangle$ with strings of $|0\rangle$'s in between (den Nijs and Rommelse, 1989),

$$O_{\mathrm{string}}^\alpha = \lim_{R\to\infty} \left\langle S_i^\alpha \left(\prod_{k=i+1}^{i+R-1} e^{i\pi S_k^\alpha} \right) S_{i+R}^\alpha \right\rangle. \tag{28}$$

The ground state of physical systems described by Hamiltonians of the form of Eq. (27) has been studied in great detail (Schollwöck *et al.*, 2004). Various phase transitions have been found between antiferromagnetic phases, Haldane phases, and a phase characterized by a large density of vanishing weights ($S_i^z=0$) along the chain.

The Affleck-Kennedy-Lieb-Tasaki (AKLT) model. Some features of the phenomenology leading to the destruction of the antiferromagnetic order can be put on a firm ground for $\beta=1/3$ (AKLT model), where the ground state of the Hamiltonian in Eq. (27) is known exactly (Affleck *et al.*, 1988). In this case it was proven that the ground state is constituted by a sea of nearest-neighbor valence bond states, separated from the first excitation by a finite gap with exponentially decaying correlation functions. Such a state is sketched in Fig. 1. In fact it is a matrix product state (MPS), i.e., it belongs to the class of states which can be expressed in the form

$$|\psi_{\mathrm{MPS}}\rangle = \sum_{s_1,\ldots,s_N}^{D} \mathrm{Tr}\, A_1^{s_1}\cdots A_N^{s_N}|s_1,\ldots,s_N\rangle, \tag{29}$$

where the matrices $(A_k^{s_i})_{lm}$ parametrize the state; $|s_i\rangle$ denotes a local basis of the D-dimensional Hilbert space; the trace contracts the indices l,m labeling bond states of the auxiliary system (namely the spin-1/2 for the AKLT model). The dimensions of A depend on the particular state considered, if the state is only slightly entangled then the dimension of A is bounded by some D_{MPS}. MPS, first discussed by Fannes *et al.* (1992), appear naturally in the density matrix renormalization group (DMRG) (Ostlund and Rommer, 1995). In one-dimensional noncritical systems they describe faithfully the ground state. In fact, as shown by Vidal, matrix product states constitute an efficient representation of slightly entangled states (Vidal, 2003).

B. Strongly correlated fermionic models

The prototype model of interacting fermions on a lattice is the Hubbard model (Essler *et al.*, 2004)

$$\mathcal{H} = -t\sum_{\langle ij\rangle}[c_{i,\sigma}^{\dagger}c_{j,\sigma} + \mathrm{H.c.}] + U\sum_{i} n_{i,\uparrow}n_{i,\downarrow} - \mu N, \tag{30}$$

where $c_{i,\sigma}$, $c_{i,\sigma}^{\dagger}$ are fermionic operators: $\{c_{i,\sigma}, c_{j,\sigma'}^{\dagger}\} = \delta_{i,j}\delta_{\sigma\sigma'}$. The coupling constant U describes the on-site repulsion and t is the hopping amplitude.

The Hubbard model possesses a $\mathrm{u}(1)\oplus\mathrm{su}(2)$ symmetry expressing the charge conservation $\mathrm{u}(1)=\mathrm{span}\{N=\Sigma_{j\sigma}n_{j\sigma}\}$ and the invariance under spin rotation $\mathrm{su}(2)=\mathrm{span}\{S^z=\Sigma_j(n_{j\uparrow}-n_{j\downarrow}), S^+=\Sigma_j c_{j,\uparrow}^{\dagger}c_{j,\downarrow}, S^-=(S^+)^{\dagger}\}$. Such a symmetry allows one to employ the total charge and magnetization as good quantum numbers. At half filling $n=N/L=1$ ($\mu=U/2$) the symmetry is enlarged to $\mathrm{so}(4)=\mathrm{su}(2)\oplus\mathrm{su}(2)$ by the generator $\eta=\Sigma_j(-)^j c_{j,\uparrow}c_{j,\downarrow}$ together with its Hermitian conjugate (Yang and Zhang, 1990). It was demonstrated that $|\Psi\rangle=(\eta)^N|gs\rangle$ are eigenstates of the Hubbard model (in any dimension), characterized by off-diagonal long-range order via the mechanism of the so-called η pairing (Yang, 1989).

In one dimension the Hubbard model undergoes a Mott transition at $U=0$ of the Berezinskii-Kosterlitz-Thouless type. By means of the Bethe ansatz solution (Lieb and Wu, 1968) it can be demonstrated how bare electrons decay in charge and spin excitations. The phenomenon of spin-charge separation occurs at low energies away from half filling. For a repulsive interaction the half-filled band is gapped in the charge sector; while spin excitations remain gapless. The mirror-inverted situation occurs for an attractive interaction where the gap is in spin excitations instead (see Essler *et al.*, 2004 for a recent review).

The Hubbard model in a magnetic field was proven to exhibit two quantum critical points at $h_c^{a\pm}=4(|U|\pm 1)$ and half filling for $U<0$, while there is one at $h_c^r=4(\sqrt{t^2+U^2}-U)$ for $U>0$ (Yang *et al.*, 2000).

If a nearest-neighbor Coulomb repulsion $V\Sigma_{\sigma,\sigma',j}n_{j\sigma}n_{j+1\sigma'}$ is taken into account in Eq. (30), a spin density wave and a charge density wave phase appear. A transition to a phase separation of high density and low density regions [see, e.g., Clay *et al.* (1999)] is also present.

The bond charge extended Hubbard model, originally proposed in the context of high T_c superconductivity (Hirsch, 1989), include further correlations in the hopping process already involved in Eq. (30). The Hamiltonian reads

$$\mathcal{H} = U\sum_{i}^{L} n_{i,\uparrow}n_{i,\downarrow} - t[1 - x(n_{i,-\sigma} + n_{i+1,-\sigma})]c_{i,\sigma}^{\dagger}c_{i+1,\sigma} + \mathrm{H.c.} \tag{31}$$

[for $x=0$ the Eq. (31) coincides with the Hubbard model (30)]. For $x\neq 0$ the hopping amplitudes are modulated by the occupancy of sites involved in the processes of tunneling. Because of the particle-hole symmetry, x can be restricted in [0,1], without loss of generality. For $x=1$ the correlated hopping term commutes with the interaction. In this case the exact ground state was shown to exhibit a variety of quantum phase transitions between insulators and superconducting regimes, controlled by the Coulomb repulsion parameter U. For $x=1$ the phase diagram is shown in Sec. IV, Fig. 15. At $U/t=4$ and $n=1$, a superconductor-insulator quantum phase transition occurs; for $-4\leqslant U/t\leqslant 4$ the ground state is characterized by off-diagonal long-range order; the low lying excitations are gapless. For $U/t=-4$ a further quantum critical point projects the ground state into the Hilbert subspace spanned by singly and doubly occupied states (Arrachea and Aligia, 1994; Schadschneider, 1995). For intermediate x the model has not been solved exactly. Numerical calculations indicate a superconducting-insulator transition controlled by U and parametrized by x. Specifically, for $0\leqslant x\leqslant 1/2$ the phase is gapped at any nonvanishing U; for $1/2<x\leqslant 1$ the onset to a superconducting phase was evidenced at some finite U (Anfossi, Degli Esposti Boschi, *et al.*, 2006).

C. Spin-boson models

A prototype model in this class is that of a quantum system coupled to a bath of harmonic oscillators [see Weiss (1999) for a review of open quantum mechanical systems] known also as the Caldeira-Leggett model. In this case the quantum system is a two level system. This class of models was investigated to study the quantum-to-classical transition and the corresponding loss of quantum coherence (Zurek, 2003).

The spin-boson Hamiltonian has the form

$$H_{sb} = -\frac{\delta}{2}S^x + \sum_n \omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right) + \frac{1}{2}S^z\sum_n \lambda_n(a_n^\dagger + a_n), \tag{32}$$

it can be demonstrated to be equivalent to the anisotropic Kondo model (Anderson *et al.*, 1970; Guinea, 1985). The coupling constants $\{\lambda_n\}$ fix the spectral density of the bath: $J(\omega)=(\pi/2)\Sigma_n\lambda_n^2\delta(\omega-\omega_n)/\omega_n$. At low energy the spectral function can be represented as a power law: $J(\omega)\propto 2\alpha\omega^s\Lambda_0^{1-s}$, where α is the parameter controlling the spin-boson interaction and Λ_0 is a ultraviolet cutoff frequency. The power s characterizes the bath. For $s=1$ the bath is called Ohmic, in this case the model has a second order quantum phase transition at $\alpha=1$ from underdamped to overdamped oscillations (where the value of the spin is frozen). The value $\alpha=1/2$ identifies a crossover regime where the two-level system is driven from coherent to incoherent oscillations. If the bath is super-Ohmic ($s>1$), the quantum critical point is washed out, while a crossover occurs at $\alpha\sim\log(\Lambda_0/\delta)$. For sub-Ohmic baths ($s<1$), several studies indicate the existence of a quantum critical point (Spohn and Dümcke, 1985). The question, however, is not completely settled (Kehrein and Mielke, 1996; Stauber and Mielke, 2002; Bulla *et al.*, 2003).

An interesting case is also that of a spin interacting with a single bosonic mode, $\lambda_n=\lambda\delta_{n,0}$:

$$H = -\frac{\delta}{2}S^x + \omega_0\left(a_0^\dagger a_0 + \frac{1}{2}\right) + \frac{\lambda_0}{2}S^z(a_0^\dagger + a_0). \tag{33}$$

Such a model describes, for example, an atom interacting with a monochromatic electromagnetic field (Cohen-Tannoudji *et al.*, 1992) via a dipole force (Jaynes and Cummings, 1963). Recently, the dynamics corresponding to Eq. (33) was studied in relation to ion traps (Cirac *et al.*, 1992) and quantum computation (Hughes *et al.*, 1998). The model defined in Eq. (33) with $S=1/2$ (Jaynes-Cummings model) was generalized and solved exactly to consider generic spin (Tavis and Cummings, 1969) in order to discuss the super-radiance phenomenon in cavity QED.

D. Harmonic lattices

The Hamiltonian for a lattice of coupled harmonic oscillators (harmonic lattice) can be expressed in terms of the phase space vector $\vec{\xi}^T=(q_1,\ldots,q_n;p_1,\ldots,p_n)$ as

$$H = \xi^T\begin{pmatrix} \frac{m}{2}\omega^2\mathbb{U} & 0 \\ 0 & \frac{1}{2m}\mathbb{1}_n \end{pmatrix}\xi, \tag{34}$$

where $\mathbb{U}$ is the $n\times n$ interaction matrix for the coordinates. If the system is translational invariant the matrix $\mathbb{U}$ is a Toeplitz matrix with periodic boundaries, also called circulant matrix (Horn and Johnson, 1994). In the case of finite range interaction of the form $\Sigma_r\Sigma_{k=1}^n K_r(q_{k+r}-q_k)^2$ and assuming periodic boundary conditions, its entries are $\mathbb{U}_{j,j}=1+2\Sigma_r\alpha_r$ and $\mathbb{U}_{j,j+r}=-\alpha_r$ with $\alpha_r=2K_r/m\omega^2$. Since the Hamiltonian (34) is quadratic in the canonical variables its dynamical algebra is $\mathrm{sp}(2n,\mathbb{R})$. Then the diagonalization can be achieved by RHR^{-1}, where $R=\otimes_{\alpha=1}^n \exp(i\theta_\alpha G_\alpha)$ with G_α the generic Hermitian element of $\mathrm{sp}(2,\mathbb{R})$.

As discussed in Sec. II.G the key quantity that characterizes the properties of harmonic systems is the covariance matrix defined in Eq. (20). For the resulting decoupled harmonic oscillators it is $\mathrm{diag}\{(m\sqrt{\eta_1}\omega)^{-1},\ldots,(m\sqrt{\eta_n}\omega)^{-1};m\sqrt{\eta_1}\omega,\ldots,m\sqrt{\eta_n}\omega)\}$, where η_j are the eigenvalues of $\mathbb{U}$. Employing the virial theorem for harmonic oscillators, the covariance matrix for a thermal state with inverse temperature $\beta=1/k_BT$ can be calculated as well,

$$V = \begin{pmatrix} (m\omega\sqrt{\mathbb{U}})^{-1}\mathbb{N}_\beta & 0 \\ 0 & (m\omega\sqrt{\mathbb{U}})\mathbb{N}_\beta \end{pmatrix}, \tag{35}$$

where $\mathbb{N}_\beta=\mathbb{1}_n+2/[\exp(\beta\omega\sqrt{\mathbb{U}})-\mathbb{1}_n]$. The range of the position or momentum correlation functions is related to the low lying spectrum of the Hamiltonian. For gapped systems the correlations decay exponentially. The absence of a gap (some eigenvalues of $\mathbb{U}$ tend to zero for an infinite system) leads to critical behavior of the system and characteristic long ranged correlations. A rigorous and detailed discussion of the relations between the gap in the energy spectrum and the properties of the correlations can be found in Cramer and Eisert (2006).

IV. PAIRWISE ENTANGLEMENT

At $T=0$ many-body systems are most often described by complex ground state wave functions which contain all correlations that give rise to the various phases of matter (superconductivity, ferromagnetism, quantum Hall systems, etc). Traditionally many-body systems have been studied by looking, for example, at their response to external perturbations, various order parameters, and excitation spectrum. The ground state of many-body systems studied with methods developed in quantum information unveil new properties. In this section we classify the ground state properties of a many-body system according to its entanglement. We concentrate on spin systems. Spin variables constitute a good example of distinguishable objects, for which the problem of entanglement quantification is most developed. We discuss various aspects starting from the pairwise entanglement, and proceed with the properties of block entropy and localizable entanglement. Most calculations are for one-dimensional systems where exact results are available. Section IV.B overviews the status in the d-dimensional case. Multipartite entanglement in the ground state will be discussed in Sec. VIII.

A. Pairwise entanglement in spin chains

1. Concurrence and magnetic order

The study of entanglement in interacting spin systems was initiated with the works on isotropic Heisenberg rings (Arnesen *et al.*, 2001; Gunlycke *et al.*, 2001; O'Connors and Wootters, 2001). O'Connors and Wootters aimed at finding the maximum pairwise entanglement that can be realized in a chain of N qubits with periodic boundary conditions. Starting from the assumption that the state maximizing the nearest-neighbor concurrence $C(1)$ was an eigenstate of the z component of the total spin (Ishizaka and Hiroshima, 2000; Munro *et al.*, 2001; Verstraete *et al.*, 2001) the problem was recast to an optimization procedure similar in spirit to the coordinate Bethe ansatz (Bethe, 1931):[4] the search for the optimal state was restricted to those cases which excluded the possibility to find two nearest-neighbor up spins. For fixed number of spins N and p spins up, the state can be written as $|\psi\rangle=\Sigma_{1\leqslant i_1<\cdots<i_p\leqslant N}b_{i_1\cdots i_p}|i_1\cdots i_p\rangle$ (b are the coefficients and i_j are the positions of the up-spins) therefore mapping the spin state onto a particle state such that the positions of the p particles correspond to those of the up spins. The maximum concurrence within this class of states can be related to the ground state gas of free spinless particles with the result

$$C(1)=-\frac{1}{N}E_{gs}=-\frac{2\sin\dfrac{\pi p}{N-p}}{\sin\dfrac{\pi}{N-p}}. \tag{36}$$

Equation (36) gives a lower bound for the maximal attainable concurrence. The isotropic antiferromagnetic chain was considered as the physical system closest to a perfectly dimerized system (classically, with alternating up and down spins). It was noticed, however, that the concurrence of the ground state of the antiferromagnetic chain is actually smaller than the value of the ferromagnetic chain, indicating that the situation is more complicated (O'Connors and Wootters, 2001). In order to clarify this point, a couple of simple examples are useful. For a system of $N=2$ spins the ground state is a singlet. However for general N (with an even number of sites) the ground state is not made of nearest-neighbor singlets [resonant valence bond (RVB) state]. For example, the $N=4$ ground state is $|gs\rangle=(1/\sqrt{6})[2|0100\rangle+2|1000\rangle-|1001\rangle-|0110\rangle-|0011\rangle-|1100\rangle]$, different from the product of two singlets. It can be seen that the effect of the last two components of the state is to reduce the concurrence with respect to its maximum attainable value. Given the simple relation Eq. (36) between the nearest-neighbor concurrence and the ground state energy, the deviation from the RVB state can be quantified by looking at the difference from the exact ground state energy corresponding to the maximum concurrence. This maximum value is reached within the set of eigenstates with zero total magnetization [the "balanced" states in O'Connors and Wootters (2001)], indicating that the concurrence is maximized only on the restricted Hilbert space of z-rotationally invariant states. Indications on how to optimize the concurrence were discussed by Meyer *et al.* (2004) and Hiesmayr *et al.* (2006). The solution to the problem for $N\rightarrow\infty$ was given by Poulsen *et al.* (2006). It turns out that the states with nearest-neighbor aligned spins [not included in O'Connors and Wootters (2001)] correspond to a "density-density" interaction in the gas of the spinless particles considered above, that hence are important for the analysis. (in the analogy of the coordinate Bethe ansatz method, this provides the "interacting picture"). Following the ideas of Wolf *et al.* (2003), the problem to find the optimum concurrence was shown to be equivalent to that of finding the ground state energy of an effective spin Hamiltonian, namely, the XXZ model in an external magnetic field. The optimal concurrence is found in the gapless regime of the spin model with a magnetization $M_z=1-2p/N$. It was further demonstrated that states considered by O'Connors and Wootters (2001) maximize the concurrence for $M_z>1/3$ (for $0\leqslant M_z\leqslant 1/3$ the states contain nearest-neighbor up spins).

The concurrence, beyond nearest neighbors, in isotropic Heisenberg antiferromagnets in an external magnetic field was discussed by Arnesen *et al.* (2001); Wang (2002a); and Fubini *et al.* (2006). The combined effect of the magnetic field and the anisotropy in Heisenberg magnets was studied by Jin and Korepin (2004a) making use of the exact results existing for the one-dimensional XXZ model. It turns out that the concurrence increases with the anisotropy Δ (Kartsev and Karshnukov, 2004). For strong magnetic fields the entanglement vanishes (the order is ferromagnetic); for large values of the anisotropy Δ the state is a classical Neel state with Ising order. Except for these cases, quantum fluctuations in the ground state lead to entangled ground states.

As discussed in Sec. III, in low-dimensional spin system there exists a particular choice of the coupling constants for which the ground state is factorized (Kurmann *et al.*, 1982). This is a special point also from the perspective of investigating the entanglement in the ground state. Several works were devoted to the characterization of the entanglement close to the *factorizing point*. It turns out that the point at which the state of the system becomes separable marks an exchange of the parallel and antiparallel sector in the ground state concurrence, see Eq. (7). As this phenomenon involves a global (long-range) reorganization of the state of the system, the range of the concurrence diverges. (We notice that several definitions of characteristic lengths associated with entanglement decay exist.) The concurrence is often observed to vanish when the two sites are farther than R

[4]Such a method relies on the existence of a "noninteracting picture" where the wave function of the system can be written as a finite sum of plane waves; the ansatz is successful for a special form of the scattering among such noninteracting pictures.

sites apart: the distance R is then taken as the range of the concurrence. For the XY model it was found that this range is

$$R \propto \left(\ln \frac{1-\gamma}{1+\gamma}\right)^{-1} \ln|\lambda^{-1} - \lambda_f^{-1}|^{-1}. \tag{37}$$

The divergence of R suggests, as a consequence of the monogamy of the entanglement (Coffman *et al.*, 2000; Osborne and Verstraete, 2006), that the role of pairwise entanglement is enhanced while approaching the separable point (Roscilde *et al.*, 2004, 2005a, 2005b). In fact, for the Ising model (i.e., $\gamma=1$), one finds that the ratio $\tau_2/\tau_1 \to 1$ when the magnetic field approaches the factorizing field h_f (Amico *et al.*, 2006). For $\gamma \neq 1$ and $h_f < h_z < h_c$ it was found that τ_2/τ_1 monotonically increases for $h_z \to h_f^+$ and that the value $(\tau_2/\tau_1)|_{h_f^+}$ increases with $\gamma \to 1$. The existence of factorizing has been also pointed out in other one-dimensional systems for both short- (Amico *et al.*, 2006; Roscilde *et al.*, 2004, 2005a) and long-range interactions (Dusuel and Vidal, 2005). In all these cases the range of the two-site entanglement diverges. The range of the concurrence was also studied for the XXZ (Jin and Korepin, 2004a) where it was shown to vary as

$$R = \left|\frac{2A(h_z)}{1-4M_z^2}\right|^{\theta}. \tag{38}$$

The exponent $\theta=2$ for finite fields, while it is $\theta=1$ for $h=0$; the coefficient $A(h)$ is known exactly in the paramagnetic phase (Lukyanov and Zamolodchikov, 1997; Lukyanov, 1999; Lukyanov and Terras, 2003) (vanishing magnetization) and in the saturation limit (Vaidya and Tracy, 1979a, 1979b). For generic h it was calculated numerically by Hikihara and Furusaki (2004). For the isotropic Heisenberg antiferromagnet, $R=1$ (Gu *et al.*, 2003).

In all previous cases the increase in the range of the pairwise entanglement means that all pairs at distances smaller than R share a finite amount of entanglement (as quantified by the concurrence). There are one-dimensional spin systems where the pairwise entanglement has qualitative different features as a function of the distance between sites. An example is the long-distance entanglement observed by Campos Venuti, Degli Esposti Buschi, and Roncaglia (2006). Given a measure of entanglement $E(\rho_{ij})$, Campos Venuti, Degli Esposti Buschi, and Roncaglia showed that it is possible that $E(\rho_{ij}) \neq 0$ when $|i-j| \to \infty$ in the ground state. Long-distance entanglement can be realized in various one-dimensional models as in the dimerized frustrated Heisenberg models or in the AKLT model. For these two models the entanglement is highly nonuniform and it is mainly concentrated in the end-to-end pair of the chain.

Spontaneous symmetry breaking can influence the entanglement in the ground state. To see this, it is convenient to introduce the thermal ground state $\rho_0 = \frac{1}{2}(|gs^o\rangle\langle gs^o| + |gs^e\rangle\langle gs^e|) = \frac{1}{2}(|gs^-\rangle\langle gs^-| + |gs^+\rangle\langle gs^+|)$ which is the $T \to 0$ limit of the thermal state. In the previous expression gs^+ and gs^- are the symmetry broken states which give the correct order parameter of the model. They are superpositions of the degenerate parity eigenstates gs^o and gs^e. Being convex, the concurrence in $gs^{\pm}$ will be larger than in $gs^{o/e}$ (Osterloh *et al.*, 2006). The opposite is true for the concave entropy of entanglement [see Osborne and Nielsen (2002) for the single spin von Neumann entropy]. The spontaneous parity symmetry breaking does not affect the concurrence in the ground state as long as it coincides with C^{I}, Eq. (7): that is, if the spins are entangled in an antiferromagnetic way (Syljuåsen, 2003b). For the quantum Ising model, the concurrence coincides with C^{I} for all values of the magnetic field, and therefore the concurrence is unaffected by the symmetry breaking, the hallmark of the present quantum phase transition. For generic anisotropies γ instead, also the parallel entanglement C^{II} is observed precisely for magnetic fields smaller than the factorizing field (Osterloh *et al.*, 2004); this interval excludes the critical point. This changes at $\gamma=0$, where the concurrence indeed shows an infinite range. Below the critical field, the concurrence is enhanced by the parity symmetry breaking (Osterloh *et al.*, 2006)

2. Pairwise entanglement and quantum phase transitions

A great number of papers have been devoted to study entanglement close to quantum phase transition (QPTs). QPT occur at zero temperature. They are induced by the change of an external parameter or coupling constant (Sachdev, 1999). Examples are transitions occurring in quantum Hall systems, localization, and the superconductor-insulator transition in two-dimensional systems. Close to the quantum critical point the system is characterized by a diverging correlation length ξ which is responsible for the singular behavior of different physical observables. The behavior of correlation functions, however, is not necessarily related to the behavior of quantum correlations present in the system. This behavior seems particularly interesting as quantum phase transitions are associated with drastic modifications of the ground state.

The critical properties in the entanglement we summarize below allow for a screening of the qualitative change of the system experiencing a quantum phase transition. In order to avoid possible confusion, it is worth noting that the study of entanglement close to quantum critical points does not provide new understanding to the scaling theory of quantum phase transitions. Rather it may be useful in a deeper characterization of the ground state wave function of the many-body system undergoing a phase transition. In this respect it is important to understand, for instance, how the entanglement depends on the order of the transition, or what is the role of the range of the interaction to establish the entanglement in the ground state. In this section we discuss exclusively the pairwise entanglement while in the

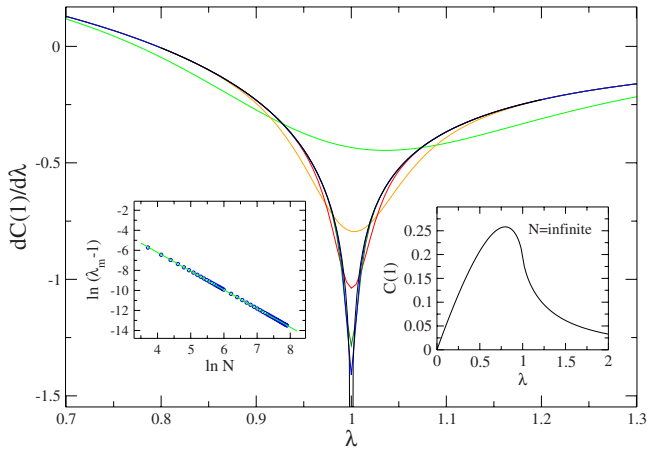

FIG. 2. (Color online) The derivative of the nearest-neighbor concurrence as a function of the reduced coupling strength. The curves correspond to different lattice sizes. On increasing the system size, the minimum gets more pronounced and the position of the minimum tends (see the left-hand side inset) towards the critical point where for an infinite system a logarithmic divergence is present. The right-hand side inset shows the behavior of the concurrence for the infinite system. The maximum is not related to the critical properties of the Ising model. From Osterloh *et al.*, 2002.

next section we approach the same problem by looking at the block entropy.[5]

Pairwise entanglement close to quantum phase transitions was originally analyzed by Osborne and Nielsen (2002), and Osterloh *et al.* (2002) for the Ising model in one dimension. Below we summarize their results in this specific case. The concurrence tends to zero for $\lambda \gg 1$ and $\lambda \ll 1$, the ground state of the system is fully polarized along the x axes (z axes). Moreover, the concurrence is zero unless the two sites are at most next-nearest-neighbors, we therefore discuss only the nearest neighbor concurrence $C(1)$ (see, however, Sec. IV.A.1 for cases where there is a longer-range pairwise entanglement). The concurrence itself is a smooth function of the coupling with a maximum close to the critical point (see the right inset of Fig. 2); it was argued that the maximum in the pairwise entanglement does not occur at the quantum critical point because of the monogamy property (it is the global entanglement that should be maximal at the critical point). The critical properties of the ground state are captured by the derivatives of the concurrence as a function of λ. The results for systems of different size (including the thermodynamic limit) are shown in Fig. 2. For the infinite chain $\partial_\lambda C(1)$ diverges on approaching the critical value as

$$\partial_\lambda C(1) \sim \frac{8}{3\pi^2} \ln|\lambda - \lambda_c|. \tag{39}$$

For a finite system the precursors of the critical behavior can be analyzed by means of finite size scaling. In the critical region the concurrence depends only on the combination $N^{1/\nu}(\lambda - \lambda_m)$, where ν is the critical exponent governing the divergence of the correlation length and λ_m is the position of the minimum (see the left inset of Fig. 2). In the case of log divergence the scaling ansatz has to be adapted and takes the form $\partial_\lambda C(1)(N,\lambda) - \partial_\lambda C(1)(N,\lambda_0) \sim Q[N^{1/\nu}\delta_m\lambda] - Q[N^{1/\nu}\delta_m\lambda_0]$, where λ_0 is some noncritical value, $\delta_m(\lambda) = \lambda - \lambda_m$, and $Q(x) \sim Q(\infty)\ln x$ (for large x). Similar results have been obtained for the XY universality class (Osterloh *et al.*, 2002). Although the concurrence describes short-range properties, nevertheless scaling behavior typical of continuous phase transition emerges.

For this class of models the concurrence coincides with C^{I} in Eq. (7) indicating that the spins can only be entangled in an antiparallel way (this is a peculiar case of $\gamma = 1$; for generic anisotropies the parallel entanglement is also observed). The analysis of the finite size scaling in the, so-called, period-2 and period-3 chains where the exchange coupling varies every second and third lattice sites, respectively, leads to the same scaling laws in the concurrence (Zhang and Burnett, 2005).

The concurrence was found to be discontinuous at the first order ferromagnetic transition $\Delta = -1$ in the XXZ chain (Gu *et al.*, 2003) [see Glaser *et al.* (2003) for explicit formulas relating the concurrence and correlators for the XXZ model in various regimes]. This result can be understood in terms of the sudden change of the wave function occurring because of the level crossing characterizing these types of quantum critical points. The behavior of the two-site entanglement at the continuous quantum critical point of the Kosterlitz-Thouless type $\Delta = 1$ separating the XY and the antiferromagnetic phases is more complex. In this case the nearest-neighbor concurrence (that is the only nonvanishing one) reaches a maximum as shown in Fig. 3. Further understanding of such behavior can be achieved by analyzing the symmetries of the model. At the antiferromagnetic point the ground state is an su(2) singlet where nearest-neighbor spins tend to form singlets; away from $\Delta = 1$, this behavior is "deformed" and the system has the tendency to reach a state of the type $\otimes|\phi_q^j\rangle$ made of q-deformed singlets corresponding to the quantum algebra $\mathrm{su}_q(2)$ with $2\Delta = q + q^{-1}$ (Pasquier and Saleur, 1990). This allows one to rephrase the existence of the maximum in the concurrence as the loss of entanglement associated to the q-deformed symmetry of the system away from $\Delta = 1$ (note that q-singlets are less entangled than undeformed ones). This behavior can be traced back to the properties of the finite size spectrum (Gu *et al.*, 2007). In fact, at $\Delta = 1$ the concurrence can be related to the eigenenergies. The maximum arises since both the transverse and longitudinal orders are power law decay-

[5]QPTs were also studied by looking at quantum fidelity (Cozzini *et al.*, 2006; Zanardi *et al.*, 2006) or the effect of single bit operations (Giampaolo, Illuminati, and Sienga, 2006; Giampaolo *et al.*, 2008).

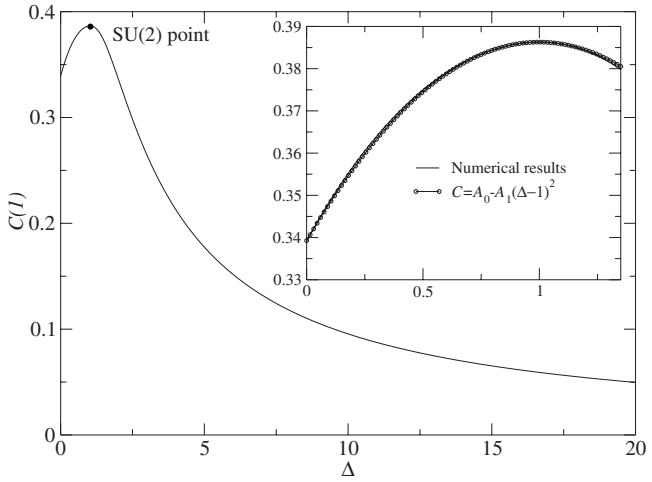

FIG. 3. Nearest-neighbor concurrence of the *XXZ* model. From Gu *et al.*, 2003.

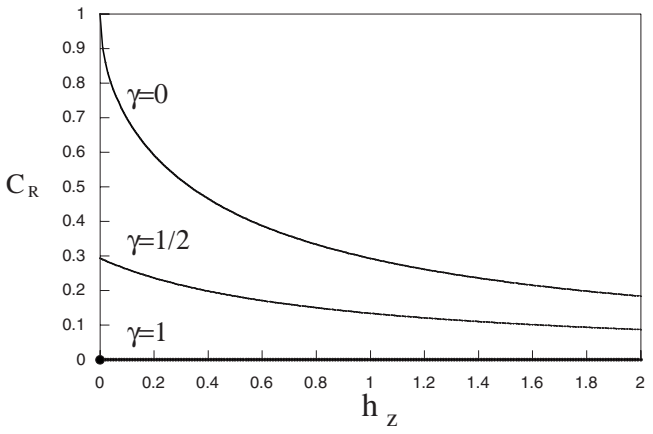

FIG. 4. The rescaled concurrence of the antiferromagnetic LMG model. The first order transition occurs at $h=0$. From Vidal, Mosseri, and Dukelsky, 2004.

ing at this critical point, and therefore the excited states contribute to $C(1)$ maximally.

Studies of finite size energy spectrum of other models like the dimerized Heisenberg chain (Sun *et al.*, 2005) and Majumdar-Ghosh model [Eq. (26) with $\alpha=1/2$] show how level crossings in the energy spectrum affect the behavior of the bipartite entanglement occurring at the quantum phase transition (Gu *et al.*, 2007).

(*a*) *LMG model*. Because of the symmetry of the LMG models [see Eq. (25)] any two spins are entangled in the same way. The concurrence C is independent on the two site indices, it can be obtained by exploiting the explicit expression of the eigenstates. Due to the monogamy of entanglement the result must be rescaled by the coordination number, $C_R=(N-1)C$ to have a finite value in the thermodynamic limit. For the ferromagnetic model (Vidal, Mosseri, and Dukelsky, 2004), it was proven that close to the continuous QPT, $\lambda=1$ characterizing the ferromagnetic LMG model, the derivative of the concurrence diverges, but, differently from Ising case, with a power law. It is interesting that C_R can be related to the so-called spin squeezing parameter $\Sigma=2\sqrt{\Delta S_{n_\perp}}$ (Wang and Burnett, 2003), measuring the spin fluctuations in a quantum correlated state (the subscript $n_\perp$ indicates a perpendicular axes to $\langle\mathbf{S}\rangle$). The relation reads $\Sigma=\sqrt{1-C_R}$. According to Lewenstein and Sanpera (1998) the two-spin reduced density operator can be decomposed into a separable part and a pure entangled state ρ_e with a certain weight Λ. Such a decomposition leads to the relation $C(\rho)=(1-\Lambda)C(\rho_e)$. Critical spin fluctuations are related to the concurrence of the pure state $C(\rho_e)$ while the diverging correlation length is related to the weight Λ (Shimizu and Kawaguchi, 2006). Analysis of critical entanglement at the first order quantum critical point of the antiferromagnetic LMG model shows that (Vidal, Mosseri, and Dukelsky, 2004) the discontinuity is observed directly in the concurrence for spin interacting with a long range; see Fig. 4.

(*b*) *Pairwise entanglement in spin-boson models*. We first discuss the Tavis-Cummings model defined in Eq. (33). In this model the spin S is proportional to the number of atoms, all interacting with a single mode radiation field. The pairwise entanglement between two different atoms undergoing the super-radiant quantum phase transition (Lambert *et al.*, 2004, 2005; Reslen *et al.*, 2005) can be investigated through the rescaled concurrence $C_N=NC$, see Fig. 5, similar to what has been discussed above for the LMG models. In the thermodynamic limit the spin-boson model can be mapped onto a quadratic bosonic system through a Holstein-Primakoff transformation (Emary and Brandes, 2003). Many of the properties of the Tavis-Cummings model bear similarities with the ferromagnetic LMG model. In the thermodynamic limit the concurrence reaches a maximum value $1-\sqrt{2}/2$ at the super-radiant quantum phase transition with a square root singularity (see also Schneider and Milburn, 2002). The relationship between the squeezing of the state and entanglement was highlighted by Sø-

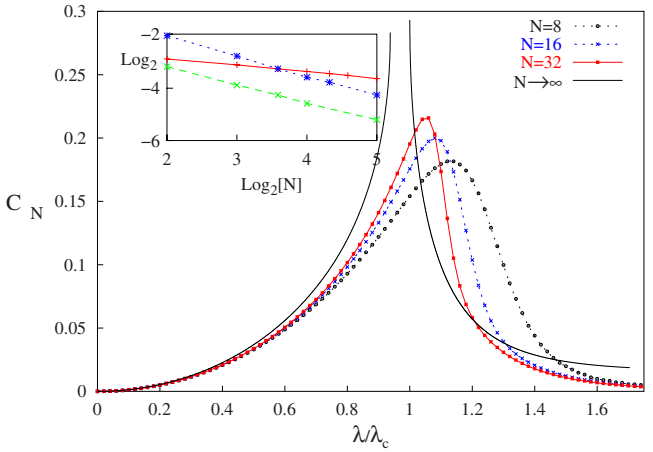

FIG. 5. (Color online) The rescaled concurrence between two atoms in the Dicke mode. The concurrence is rescaled both for finite N and in the thermodynamic limit. The inset shows the finite size scaling. From Lambert *et al.*, 2004.

rensen and Mølmer (2001) and analyzed in more details by Stockton *et al.* (2003) where it was also suggested how to deal with entanglement between arbitrary splits of symmetric Hilbert spaces (such as the Dicke state span).

Entanglement between the qubits and a single mode and between two spins with a Heisenberg interaction of the XXZ type, additionally coupled to a single bosonic field, was considered by Liberti *et al.* (2006a, 2006b) and He *et al.* (2006), respectively.

3. Entanglement versus correlations in spin systems

From the results summarized above it is clear that the anomalies characterizing the quantum critical points are reflected in the two-site entanglement. At a qualitative level this arises because of the formal relation between the correlation functions and the entanglement. A way to put this observation on a quantitative ground is provided by a generalized Hohenberg-Kohn theorem (Wu *et al.*, 2006). Accordingly, the ground state energy can be considered as a unique function of the expectation values of certain observables. These, in turn, can be related to (the various derivatives of) a given entanglement measure (Wu *et al.*, 2004; Campos Venuti, Degli Esposti Boschi, *et al.*, 2006).

Specifically, for a Hamiltonian of the form $H=H_0+\Sigma_l\lambda_l A_l$, with control parameters λ_l associated with operators A_l, it can be shown that the ground state reduced operators of the system are well-behaved functions of $\langle A_l\rangle$. Then, any entanglement measure related to reduced density operators $M=M(\rho)$ is a function of $\langle A_l\rangle$ (in absense of ground state degeneracy) by the Hellmann-Feynman theorem: $\partial E/\partial\lambda_l=\langle\partial E/\partial\lambda_l\rangle=\langle A_l\rangle$. Therefore it can be proven that

$$M(\langle A_l\rangle)=M\left(\frac{\partial E}{\partial\lambda_l}\right), \tag{40}$$

where E is the ground state energy. From this relation emerges how the critical behavior of the system is reflected in the anomalies of the entanglement. In particular, first order phase transitions are associated with the anomalies of M while second order phase transitions correspond to a singular behavior of the derivatives of M. Other singularities, like those in the concurrence for models with three-spin interactions (Yang, 2005), are due to the nonanalyticity intrinsic in the definition of the concurrence as a maximum of two analytic functions and the constant zero.

The relation given in Eq. (40) was constructed explicitly for the quantum Ising, XXZ, and LMG models (Wu *et al.*, 2006). For the Ising model: $\Sigma_l\lambda A_l=h\Sigma_l S_l^z$; the divergence of the first derivative of the concurrence is then determined by the nonanalytical behavior of $\langle S^xS^x\rangle$ (Wu *et al.*, 2004). For the XXZ model: $\Sigma_l\lambda_l A_l=\Delta\Sigma_l S_l^zS_{l+1}^z$. At the transition point $\Delta=1$ both the purity and the concurrence display a maximum. It was proven that such a maximum is also reflected in a stationary point of the ground state energy as a function of $\langle S_i^zS_{i+1}^z\rangle$; the concurrence is continuous since the Berezinskii-Kosterlitz-Thouless transition is of infinite order. A relevant caveat to Eq. (40) is constituted by the uniaxial LMG model in a transverse field (with $h_y=0$ and $\gamma=0$) that displays a first order QPT for $h_x=0$. The concurrence is continuous at the transition since it does not depend on the discontinuous elements of the reduced density matrix (Vidal, Palacios, and Mosseri, 2004).

The relation between entanglement and criticality was also studied in the spin-1 XXZ with single ion anisotropy. It was established that the critical anomalies in the entropy experienced at the Haldane large D [if an axial anisotropy $D\Sigma_i(S_i^z)^2$ is added to the Hamiltonian in Eq. (27)] transition fans out from the singularity of the local order parameter $\langle(S^z)^2\rangle$ (Campos Venuti, Degli Esposti Boschi, *et al.*, 2006a).

A way to study the general relation between entanglement and critical phenomena was pursued by Haselgrove *et al.* (2004). It was argued how for systems with finite range interaction a vanishing energy gap in the thermodynamic limit is an essential condition for the ground state to have nonlocal quantum correlations between distant subsystems.

4. Spin models with defects

The problem of characterizing entanglement in chains with defects was addressed first for the quantum XY models with a single defect in the exchange interaction term of the Hamiltonian (Osenda *et al.*, 2003). It was found that the effect of the impurity is to pin the entanglement. Moreover, the defect can induce a pairwise entanglement on the homogeneous part of the system that was disentangled in the pure system. Even at the quantum critical point the finite size scaling of the critical anomaly of the concurrence is affected by the distance from the impurity. This basic phenomenology was observed in a variety of different situations that we review below.

The presence of two defects has been analyzed in the XXZ chain. It turns out (Santos, 2003) that various types of entangled states can be created in the chain by spin flip excitations located at defect positions. The entanglement oscillates between defects with a period that depends on their distance. The anisotropy Δ of the chain is a relevant parameter controlling the entanglement between defects. Small anisotropies can suppress the entanglement (Santos and Rigolin, 2005). This kind of localization, which can be exploited for quantum algorithms, was studied by Santos *et al.* (2005). The entanglement was also studied in systems with defects in the presence of an external magnetic field (Apollaro and Plastina, 2006). It was demonstrated that such a defect can lead to an entanglement localization within a typical length which coincides with the localization length.

A possible way to mimic a defect is to change the boundary conditions. The concurrence was studied for the ferromagnetic spin-1/2 XXZ chain with an antiparallel boundary magnetic field which gives rise to a term in the Hamiltonian of the form $H_{\text{boundary}}=h(S_1-S_N)$ (Alcaraz *et al.*, 2004). The boundary field triggers the pres-

ence of domain walls in the system that induces a first order phase transition between ferromagnetic and kink-type phases at $h_c=\sqrt{\Delta^2-1}$. In the ferromagnetic phase the pairwise entanglement vanishes. In the kink-type phase the concurrence acquires a finite value (for $h=0$ the ground state is factorized). For a finite chain of length L pairwise entanglement is generically triggered from the external magnetic field, and it is enhanced at the center of the chain. In the gapless and antiferromagnetic regimes oscillation in nearest-neighbor entanglement is established in the system resulting from the tendency to reach the antiferromagnetic order. The oscillations are more pronounced in the gapped phase. Finally, a critical inflection point was noticed in the measure of Meyer and Wallach (2002) for the global entanglement at the transition point. The spin-1/2 XXX antiferromagnetic chain with open boundary conditions with single defect was studied by Wang (2004). It was proven that a threshold value exists for coupling between the impurity and the rest of the system at which the concurrence between them is switched on; for smaller values the entanglement is dimerized in such a way that the monogamy property prevents the impurity to be entangled with the rest of the chain.

The case of many defects was also studied. For the quantum Ising model it was found (Huang *et al.*, 2004) that disorder can shift the point at which the concurrence is maximum, eventually washing out the critical behavior (strong disorder). The concurrence tends to be suppressed at the lattice site corresponding to the center of the Gaussian; such effect is more robust for near critical chains. Quantum XY and XXZ chains with Gaussian disorder in the exchange interaction have been also studied to investigate how the quantum criticality of the concurrence is robust by insertion of the inhomogeneities in the chain (Cai *et al.*, 2006; Hoyos and Rigolin, 2006).

B. Two- and three-dimensional systems

In higher dimensions nearly all results were obtained by means of numerical simulations. The concurrence and localizable entanglement in two-dimensional quantum XY and XXZ models (Syljuåsen, 2003a) were considered. The calculations are based on quantum Monte Carlo (QMC) simulations and the use of stochastic series expansion for spin systems (Sandvik and Kurkij, 1991; Syljuåsen and Sandvik, 2002). Although the concurrence for the 2D models results to be qualitatively similar to the one-dimensional case, it is much smaller in magnitude. It is the monogamy that limits the entanglement shared among the number of neighbor sites (which is larger in two dimensions as compared with chains). Finally, it is observed that the maximum in the concurrence occurs at a position closer to the critical point than in the 1D case.

By studying appropriate bounds (concurrence of assistance and the largest singular value of the connected correlation functions), it was proven for the XXZ model that the localizable entanglement is long ranged in the XY region up to the isotropic antiferromagnetic point. Similar to the case of the quantum XY chain, the bounds for the localizable entanglement are very tight in this case.

The pairwise entanglement in the d-dimensional XXZ model was studied by Gu *et al.* (2005). The concurrence reaches its highest value at the antiferromagnetic quantum critical point $\Delta=-1$. A spin-wave analysis corroborated by numerical exact diagonalization indicates that the concurrence develops a cusp in the thermodynamic limit, only for $d\geqslant 2$. Such behavior can be explained by noting that the level crossing between the ground and first excited states occurring at the antiferromagnetic point causes a nonanalyticity in the ground state energy. The enhanced pairwise entanglement at the antiferromagnetic point together with its nonanalyticity support the conjectured existence of long-range order for two-dimensional antiferromagnets. Further support to this conjecture is the strong size dependence of the von Neumann entropy that becomes singular in thermodynamic limit (Gu *et al.*, 2006).

The ground state entanglement in the two-dimensional XYZ model was analyzed by Roscilde *et al.* (2005b) with quantum Monte Carlo simulations. The divergence of the derivative of the concurrence at the continuous phase transition, observed in $d=1$, was confirmed; in this case the range of the pairwise entanglement extends only to a few lattice sites. By studying the one and the two tangle of the system, it was proven that the QPT is characterized by a cusp minimum in the entanglement ratio τ_1/τ_2. The cusp is ultimately due to the discontinuity of the derivative of τ_1. The minimum in the ratio τ_1/τ_2 signals the enhanced role of the multipartite entanglement in the mechanism driving the phase transition. Moreover, by looking at the entanglement it was found that the ground state can be factorized at a certain value of the magnetic field. The existence of the factorizing field in $d=2$ was proven for any 2D XYZ model in a bipartite lattice. Unexpectedly the relation implying the factorization is very similar to that one found in $d=1$.

Multiple spin exchange is believed to play an important role in the physics of several bidimensional magnets (Schollwöck *et al.*, 2004). Entanglement in the ground state of a two leg ladder with four spin ring exchange was evaluated by means of the concurrence (Song *et al.*, 2006).

C. Pairwise entanglement in fermionic models

1. Noninteracting fermions

The site-based entanglement of spin degrees of freedom through the Jordan-Wigner transformation has been exploited for calculating the concurrence of nearest-neighbor sites and the single site von Neumann entropy (see Sec. II.F) for the one-dimensional tight-binding model in the presence of a chemical potential for spinless fermions by Zanardi (2002). This model is

related to the isotropic XX model in a transverse magnetic field. In this specific case, no double occupancy can occur and the concept of entanglement coincides with that for spin 1/2. It was found that the nearest-neighbor concurrence of the ground state at $T=0$ assumes its maximum at the half-filled chain. Due to particle-hole symmetry, the concurrence results are symmetric with respect to half filling. At finite temperatures it was found that the threshold temperature for vanishing concurrence is independent of the chemical potential. Raising the chemical potential leads from a monotonically decreasing concurrence with raising temperature at low filling fraction to the formation of a maximum at a certain temperature for high filling fractions.

The continuous limit of the tight-binding fermion model is the ideal Fermi gas. In this system, the spin entanglement between two distant particles has been studied by Vedral (2003). There, depending on the dimensionality, the pairwise spin entanglement of two fermions was found to decrease with their distance for a finite range R of the concurrence. The two spin reduced density matrix is

$$\rho_{12}=\frac{1}{4-2f^2}\begin{pmatrix}1-f^2 & 0 & 0 & 0\\ 0 & 1 & -f^2 & 0\\ 0 & -f^2 & 1 & 0\\ 0 & 0 & 0 & 1-f^2\end{pmatrix}, \tag{41}$$

where $f(x)=d[J_1(x)/x]$ with $d\in\{2,3\}$ the space dimension and J_1 the (spherical for $d=3$) Bessel function of the first kind (Vedral, 2003; Oh and Kim, 2004). This density matrix is entangled for $f^2\geqslant 1/2$. As a consequence, there is spin entanglement for two fermions closer than $d_0\approx 0.65\pi/k_f$ for $d=3$ and $d_0\approx 0.55\pi/k_f$ for $d=2$ (k_f is the Fermi momentum). A finite temperature tends to diminish slightly the range of pairwise spin entanglement (Oh and Kim, 2004).

It should not be surprising that noninteracting particles are spin entangled up to some finite distance. It is true that the ground state and even an arbitrary thermal state of noninteracting fermions has vanishing entanglement among particles [which should not be confused with the nonvanishing entanglement of particles (Dowling, Doherty, and Wiseman, 2006)], since the corresponding states are (convex combinations of) antisymmetrized product states. However, disentanglement in momentum space typically leads to entanglement in coordinate space. A monochromatic plane wave of a single particle, for example, corresponds to a W state, which contains exclusively pairwise entanglement in coordinate space for an arbitrary distance of the sites. Furthermore, a momentum cutoff at k_f corresponds to a length scale of the order k_f^{-1}.

It is interesting that *fuzzy* detection of particles in coordinate space increases the entanglement detected by the measurement apparatus. To this end, Cavalcanti *et al.* (2005) calculated the two-position reduced density matrix defined by $\rho^{(2)}_{ss',tt'}=\langle\Psi_{t'}(r')^\dagger\Psi_t(r)^\dagger\Psi_{s'}(r')\Psi_s(r)\rangle$ with blurred field operators $\Psi_s(r):=\int dr'dp\psi_s(p)D(r-r')e^{ipr'}$, where $D(r-r')=(1/\sqrt{2\pi}\sigma)\exp(-|r-r'|/2\sigma^2)$ is a Gaussian distribution describing the inaccuracy of the position measurement. This could be understood from the blurred field operators being coherent sums of local field operators; the entanglement measured by the apparatus as described above then is the bipartite entanglement between two regions of width σ around r and r'. This entanglement is larger than the average of all pairwise contributions out of it due to the superadditivity of the entropy or negativity. An analysis by Vedral (2004b) for the three fermion spin density matrix revealed that the state carries entanglement within the W class (Dür *et al.*, 2000), provided the three particles are in a region with radius of the order of the inverse Fermi momentum; a similar reasoning applies to n fermions in such a region (Vedral, 2004b; Lunkes *et al.*, 2005).

2. Pairing models

Itinerant systems, where the focus of interest is the entanglement of degrees of freedom forming a representation of su(2) in terms of the fermionic operators, have been the subject of intense investigation. This line has been followed by Shi (2004) and Zanardi (2002) for analyzing a connection to BCS superconductivity and to the phenomenon of η pairing, a possible scenario for high T_c superconductivity [see also Fan *et al.* (2004) and Vedral (2004a, 2004b)]. Such states appear as eigenstates of the Hubbard model with off-diagonal long-range order (see Sec. III.B). A simplified model of BCS-like pairing for spinless fermions has been studied by Zanardi (2002). The concurrence of the two qubits represented by the modes k and $-k$ has been found to be a monotonically increasing function of the order parameter; it drops to zero significantly before the critical temperature is reached. For electrons with spin, a connection between the BCS order parameter and the local von Neumann entropy in the particle number projected BCS ground state has been proposed by Shi (2004) [see also Gedik (2002)].

States with off-diagonal long-range order by virtue of η pairing are defined in Sec. III.B. These are symmetric states and their concurrence vanishes in the thermodynamic limit due to the sharing property of pairwise entanglement of su(2) degrees of freedom. Consequently, a connection to the order parameter of off-diagonal long-range order $O_\eta=\langle\Psi|\eta_j^\dagger\eta_k|\Psi\rangle=N(L-N)/L(L-1)\to n(1-n)$ (with $N,L\to\infty$ and fixed filling fraction n) cannot be established, not even for the rescaled concurrence, since $C\to 1/L$ (see also the analysis for the LMG model in Sec. IV.A.2). Nevertheless, the state is entangled, as can be seen from the entropy of entanglement and the geometric measure of entanglement (Wei and Goldbart, 2003). The latter is tightly connected to the relative entropy (Wei *et al.*, 2004). Both have been calculated by Vedral (2004a) and indicate the presence of multipartite entanglement.

3. Kondo models

The Kondo models are paradigms to explore quantum impurity problems. They identify a special class of physical systems whose macroscopic properties are dramatically influenced by the presence of few impurities with quantum internal degrees of freedom (Hewson, 1997). In its simplest formulation, the effective Hamiltonian describes a single impurity spin interacting with a band of free electrons. The many-body screening of the impurity spin provided by electrons as a collective effect leads the system from weak coupling to strong coupling regimes (Andrei, 1995); the length scale of the screening cloud is $\xi_K = v/T_K$, where v is the speed of the low lying excitations and T_K is the Kondo temperature.

In the first studies of the entanglement the charge degrees of freedom of electrons were considered frozen. The pairwise entanglement of spin degrees of freedom in the isotropic Kondo model was analyzed (Oh and Kim, 2006) within the variational formalism of Yosida where the Kondo singlet is described as $|\Psi_s\rangle = (1/\sqrt{2})(|\phi_\downarrow\rangle|\chi_\uparrow\rangle - |\phi_\uparrow\rangle|\chi_\downarrow\rangle)$, with $|\chi_\sigma\rangle$ the impurity spin states and $|\phi_\sigma\rangle$ the electronic states with an unbalanced spin σ (Hewson, 1997). It is commonly known that the reduced density operator of the impurity is found to be maximally mixed, meaning that the Fermi sea and the impurity spin are in a maximally entangled state (the Kondo singlet). The impurity spin and a single electron are in a Werner state by superposition of the background and Kondo singlets. Due to the entanglement monogamy (electrons cooperatively form a singlet with spin) two electrons cannot be entangled with each other within the Kondo cloud and the single-electron spin entanglement vanishes in the thermodynamical limit. Pairs of electronic spin can be nevertheless entangled in a finite system through the scattering with the spin impurity; this effect can be used to manipulate the electron-electron entanglement by performing a projective measure on the impurity spin (Yang *et al.*, 2007). This suggests that some amount of electron-electron entanglement might be extracted even in the thermodynamical limit where it was demonstrated that the Kondo resonance is washed out by measurement (Katsnelson *et al.*, 2003)—effectively removing the constraint of the entanglement sharing.

The two impurity Kondo model was also studied. The new feature here is the Ruderman-Kittel-Kasuya-Yosida (RKKY) effective interaction between the impurity spins $\mathbf{S_1}$ and $\mathbf{S_2}$ that competes with the Kondo mechanism (favoring nonmagnetic states); it is ferromagnetic or antiferromagnetic depending on the distance between the impurities. Because of such interplay a quantum critical point emerges in the phase diagram separating the spin-spin interaction regime from the phase where the two spins are completely screened (Jones *et al.*, 1988; Affleck and Ludwig, 1992).

As for the single impurity, the two impurity spins are in a Werner state, for which the concurrence is characterized by a single parameter p_s, exhibiting a singlet type of entanglement between the two spin impurities. The concurrence is found to vanish at the critical point.

For a ferromagnetic RKKY interaction the concurrence between the impurity spin vanishes identically as the result of a $S=1$ Kondo screening. It turns out that (Cho and McKenzie, 2006) the impurity spins can be entangled (with a finite concurrence) by the RKKY interaction only when a certain amount of antiferromagnetic correlations $f_s = \langle \mathbf{S_1 S_2} \rangle$ is established in the system; the value of the correlation function reached is at the quantum critical point. The entanglement between the conduction electrons and Kondo impurities is quantified by a combined analysis of the von Neumann entropies of the two impurities and single impurity (tracing out both electronic spins and the remaining impurity). The latter quantity is maximized independent on f_s, meaning that the impurity spin is completely screened either by the Kondo cloud or by the other impurity spin. In the regime where the Kondo mechanism dominates, the concurrence cannot be finite because of the entanglement sharing.

Entanglement in the Kondo physics of double quantum dots in an external magnetic field was studied by Ramsak *et al.* (2006). The main phenomenology results to be consistent with the scenario depicted by Cho and McKenzie (2006) especially if the dots are arranged in series (each dot is coupled to the leads exclusively, resembling the configuration of the Kondo spins embedded in electrons). The concurrence switches to finite values for a certain threshold of the interdot coupling (for which the assumptions of negligible charge fluctuations results are still valid). The temperature weakens the entanglement between the qubits at $T > T_K$. For the side- and parallel-coupled dots a more intense coupling among the qubits is required to entangle them. For the side-coupled dots this results because one has to win on the enhanced Kondo effect on the dot coupled to the leads ("two stage Kondo" effect); therefore the critical interdots coupling is $\sim T_K$. For the parallel-coupled arrangement the concurrence is zero because the effective RKKY interaction turns out to be ferromagnetic up to a certain value that is the threshold to entangle electrons.

The RKKY interaction effectively controls the entanglement among the qubits in the case of many impurity spins arranged as in the Kondo necklace model (Saguia and Sarandy, 2003). The Hamiltonian describes a Kondo lattice where the localized impurity spins, displaced in every lattice site, interact with the (pseudo) spins of the electrons [see Tsunetsugu *et al.* (1997) for a review on Kondo lattice models]. This results in the fact that additional on-site spin-spin interaction imposes a "selective" monogamy of the entanglement, depending on whether the Heisenberg interactions are ferromagnetic or antiferromagnetic. The effects of finite temperature and magnetic field were also considered in anisotropic models. In these models a critical field emerges that exists separating different patterns in the thermal entanglement between the eigenstates of the model.

Fluctuations of the charge degrees of freedom of electrons (frozen in the references cited above) in the RKKY mechanism was discussed quantifying the en-

tanglement of particles in a small cluster described by the periodic Anderson model (Samuelsson and Verdozzi, 2007). It was evidenced that the ground state of the system is characterized by a double occupancy of electronic levels, whose entanglement can be only partially captured by assuming them as simple qubits.

D. Entanglement in itinerant bosonic systems

In contrast to the free Fermi gas, in bosonic systems the phenomenon of Bose-Einstein condensation (BEC) takes place at sufficiently low temperatures. Then, a macroscopic portion of bosons is found in the single particle ground state of the system. This state is a symmetrization of a product state in momentum space, and in fact all eigenstates of the ideal Bose gas are of that structure. Nevertheless, in principle entanglement could be present when going in the coordinate basis. It results, however, that this is not the case. Neither do two distant bosons carry spin entanglement (Vedral, 2003) nor are two distant groups of n and m particles entangled (Dowling, Doherty, and Wiseman, 2006) when superselection rules for the particle number are applied in both regions (Wiseman and Vaccaro, 2003). Simon (2002) employed a different notion of entanglement: the state of either part of a certain bipartition of the BEC has been viewed as a *qudit*, or, more precisely, an N-level system, where N is the number of bosons in the condensate and the different states in both regions are labeled by its occupation number. The entropy of entanglement for a spatial bipartition of the BEC is then nonzero. A proposal for entangling internal atomic degrees of freedom in a weakly interacting BEC has been put forward by Helmerson and You (2001); and Sørensen *et al.* (2001).

The disentangled modes in a BEC naturally become entangled by means of interactions or scattering between these modes. This has been exemplified by Vedral (2003) [see also Shi (2004)] for when the scattering strength is independent of the momentum transfer q: $H=\Sigma_q \varepsilon(p) a_p^\dagger a_p + V \Sigma_{p,p',q} a_{p+q}^\dagger a_{p'-q}^\dagger a_{p'} a_p$. The Hamiltonian can be diagonalized for $q=0$ and $p'=-p$ by means of a Bogoliubov transformation $a_p =: u_p b_p + v_p b_{-p}^\dagger$. We observe that such transformation entangles the two modes $\pm p$. The corresponding entropy of entanglement is (Vedral, 2003) $S_{p,-p}=-(v_p/u_p)^2 \ln (v_p/u_p)^2 - [1-(v_p/u_p)^2]\ln[1-(v_p/u_p)^2]$. If $u_p=v_p$, the reduced state for mode p is maximally mixed, and hence the modes p and $-p$ are maximally entangled. The entanglement entropy for a bipartition in positive and negative modes is then given by $\Sigma_{p>0} S_{p,-p}$. This constitutes a simple example on how mode mixing generates entanglement. Such a scenario is rather generic; a curious example is the entanglement of the accelerated vacuum and from the viewpoint of relatively accelerated observers due to the Unruh effect [see, e.g., Alsing and Milburn (2003); Vedral (2003); Benatti and Floreanini (2004); Fuentes-Schuller and Mann (2005)].

Further studies of entanglement in bosonic systems include the analysis in two-mode condensates (Hines *et al.*, 2002), in dipolar-couplar (Ng and Burnett, 2007), and in two-species spinor Bose condensates (Shi and Niu, 2006).

E. Entanglement of particles

Studies which use measures for indistinguishable particle entanglement (see Sec. II.F) in the area of many-body systems are still only few, particularly regarding the use of the fermionic concurrence, accounting for the possibility of double occupancy (with spin degrees of freedom). The entanglement of particles and its difference with the usual spin entanglement has been discussed by Dowling, Doherty, and Wiseman (2006), starting with very small systems as two spinless fermions on four lattice sites and the Hubbard dimer, and then for the tight binding model in one spatial dimension, in order to compare with existing results for the spin entanglement by Vedral (2003). For the Hubbard dimer (a two-site Hubbard model), these authors compare their results with those for the entanglement measured by the local von Neumann entropy without superselection rule for the local particle numbers (Zanardi, 2002). Whereas the latter signals decreasing entanglement in the ground state with increasing U/t, the entanglement of particles increases (Dowling, Doherty, and Wiseman, 2006). This demonstrates that imposing superselection rules may lead to qualitatively different behavior of the entanglement. Interestingly, an increase with U/t is also observed for the entanglement of modes without imposing superselection rules (Deng *et al.*, 2005).

We finish this section by noting a recent proposal of an experiment in order to decide whether even entanglement merely due to the statistics of the indistinguishable particles can be useful for quantum information processing (Cavalcanti *et al.*, 2006).

V. ENTANGLEMENT ENTROPY

An important class of works analyzing the entanglement in a many-body system considered a bipartition of the system dividing it into two distinct regions A and B as shown in Fig. 6. If the total system is in a pure state then a measure of the entanglement between A and B is given by the von Neumann entropy S associated with the reduced density matrix of one of the two blocks ($\rho_{A/B}$).

Motivated by the work of Fiola *et al.* (1994) and Holzhey, Larsen, and Wilczek (1994) born in the context of black hole physics, the problem was first reanalyzed in the framework of quantum information by Vidal and co-workers for quantum spin chains (Vidal *et al.*, 2003) and by Audenaert and co-workers for harmonic lattices (Audenaert *et al.*, 2002).

In studying the properties of block entropy it is important to understand its dependence on the properties (topology, dimensions, etc.) of two regions A and B. A key property which is explored to understand its range of validity is known as the area law (Srednicki, 1993). When it holds the reduced entropy S would depend only

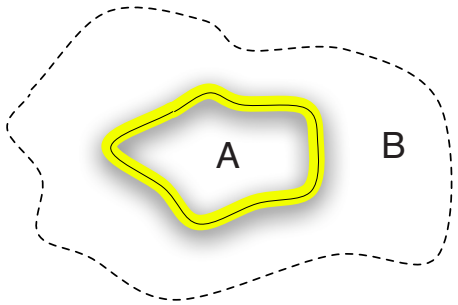

FIG. 6. (Color online) The block entropy is evaluated after partitioning the system in regions A and B. For finite range correlations, it is intuitive that the wave function of the system is factorized $|\Psi_A\rangle|\Psi_B\rangle$ by removing the region at the boundary. Accordingly the reduced entropy would vanish.

on the surface of separation between regions A and B. In a d-dimensional system this means $S_l \sim \ell^{d-1}$, where ℓ is the order of the size of one block [see Riera and Latorre (2006) for a discussion of the area law]. In the rest of the section we discuss several different physical systems in one- and higher-dimensional lattices to see when the area law holds. Here we consider only many-body systems in their ground state, the thermal effect and the dynamical properties of the entropy will be discussed in Secs. VII and IX, respectively.

The entanglement entropy is not a mere theoretical concept but might be measured. Following the procedure proposed by Klich *et al.* (2006) the measure of S can be related to the distribution measurement of suitably chosen observables.

A. One-dimensional spin systems

We start our review on the properties of the block entropy by analyzing the case of one-dimensional spin systems to which a large body of work has been devoted. By means of the Jordan-Wigner transformation it is possible to map the models onto lattice fermions, hence the results discussed here are applicable to fermionic models (after the appropriate mapping) as well. A particularly important case which is amenable of an exact solution is the XY model (see Sec. III.A.1) which can be mapped onto a free fermion model. For this case we discuss in more detail the method to calculate the block entropy. In this section we consider only chains with short-range interaction.

1. Spin chains

In one dimension the surface separating the two regions is constituted by two points, therefore the area's law would imply that the reduced entropy is independent of block size. This is indeed the case when the system is gapped and hence the correlation length ξ is finite [see Hastings (2004) for a rigorous proof]. In the gapless case, $\xi=\infty$, logarithmic corrections appear and the prefactor of the block entropy is *universal*, related to the central charge of the underlying conformal field theory. Holzey *et al.* (1994), benefiting from earlier work of Cardy and Peschel (1988), analyzed the block entropy of a $(1+1)$-dimensional massless bosonic field. Vidal *et al.* studied numerically one-dimensional Ising and Heisenberg chains (Vidal *et al.*, 2003; Latorre *et al.*, 2004) and conjectured that the block entropy would saturate for a noncritical chain while diverge logarithmically with a prefactor related to the central charge of the underlying conformal theory (Holzhey *et al.*, 1994). Such violation of the area law in critical systems reflects how the mixedness of the state increases by the partial tracing operation, regardless of the spatial extension of the spin block.

Calculation of the block entropy by means of conformal field theory, generalizing in several respects the results of Holzey *et al.* by including the case of free and periodic boundary conditions, different partitions, noncritical systems, and finite temperature, has been performed by Calabrese and Cardy (2004). Starting from the work on the XX model of Jin and Korepin (2004b), important explicit analytic calculations for a number of one-dimensional XY spin (free fermion) models have been carried out by Jin and Korepin (2004b); Korepin (2004); Peschel (2004, 2005); Eisler and Zimborás (2005); Its *et al.* (2005, 2007); Keating and Mezzadri (2005); Popkov and Salerno (2005); Franchini *et al.* (2006, 2007); Weston (2006). Numerical calculations on the XX and XXZ models were also performed by De Chiara *et al.* (2005) and Laflorencie (2005). The study of entanglement entropy is of great interest in a more general context [see Fiola *et al.* (1994); Holzhey *et al.* (1994); Casini and Huerta (2005); Casini *et al.* (2005); Ryu and Takayanagi (2006a, 2006b) and references therein], Casini and co-workers, for example, evaluated the entanglement entropy for both massive scalar field theory (Casini and Huerta, 2005) and Dirac fields (Casini *et al.*, 2005)]. Recently the area law for gapped one-dimensional systems was proven by Hastings (2007a, 2007b).

The main features of the reduced entropy in one-dimensional spin Fermi systems can be summarized as follows (for clarity we discuss only the long distance behavior as dictated by the underlying conformal field theory).

- At criticality a one-dimensional system has a block entropy which diverges logarithmically with block size. If the block is of length ℓ and the system is L long with periodic boundary condition then S_ℓ is given by

$$S_\ell = \frac{c}{3}\log_2\left[\frac{L}{\pi a}\sin\left(\frac{\pi}{L}\ell\right)\right] + A, \tag{42}$$

  where c is the central charge of the underlying conformal field theory and a is an ultraviolet regularization cutoff (for example, the lattice spacing in spin systems). A is a nonuniversal constant. For the Ising model $c=1/2$ while for the Heisenberg model $c=1$ (see Fig. 7).
- Slightly away from criticality, where the system has a large but finite correlation length $\xi \gg a$ and the

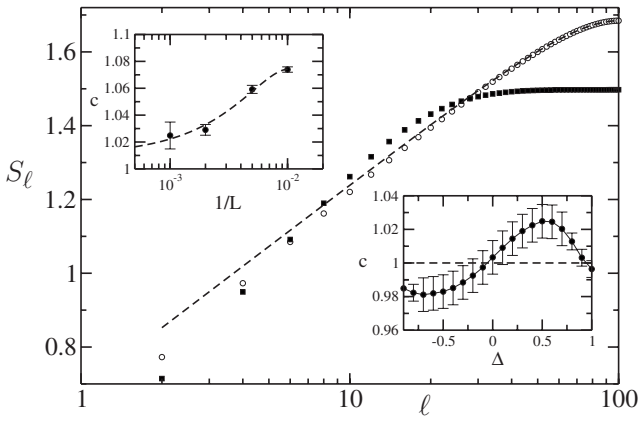

FIG. 7. The block entropy S_ℓ for $L=200$ in the case of an XXZ Heisenberg chain for a critical value $\Delta=0.0$ (circles) and noncritical value $\Delta=1.8$ (squares). The critical data compared with the conformal field theory prediction (dashed line). Lower inset: Central charge extrapolated by fitting the numerical data S_ℓ with Eq. (42) (with a factor $1/2$ as done in the numerical calculation with the block at the boundary) for different values of Δ. The data are for $L=1000$. Upper inset: Scaling of c extrapolated as a function of $1/L$ for the worst case $\Delta=0.5$ and compared to a quadratic fit (dashed line). From De Chiara *et al.*, 2005.

Hamiltonian is short ranged, the block entropy saturates to a finite value (see Fig. 7)

$$S_\ell \sim \frac{c}{3}\log_2\frac{\xi}{a} \quad \text{for } \ell\to\infty. \tag{43}$$

- An extension to finite temperature in the critical case has been obtained by means of conformal field theory (Calabrese and Cardy, 2004) and by conformal mapping together with the second law of thermodynamics (Korepin (2004) with the result

$$S_\ell = \frac{c}{3}\log_2\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi}{\beta}\ell\right)\right] + A, \tag{44}$$

where A is a constant and β the inverse temperature. In the finite temperature case, however, the block entropy is not a measure of the entanglement between the two partitions as the state to start with is mixed. A comparison of numerical data with the CFT prediction is shown in Fig. 7 for the XXZ model.

By now it is clear that the various measures of entanglement are sensitive to the presence of quantum phase transitions, the scaling of the entropy gives excellent signatures as well. Recent work tried to construct efficient ways to detect quantum phase transitions by analyzing the reduced entropy for small clusters. One-site entropy has been considered by Gu *et al.* (2004) and Chen *et al.* (2006). Chen *et al.* analyzed the entanglement of the ground states in XXZ and dimerized Heisenberg spin chains as well as in a two-leg spin ladder suggesting that the phase boundaries might be identified based on the local extreme of the entanglement entropy (Chen *et al.*, 2006). Legeza and co-workers (Legeza and Sólyom, 2006; Legeza *et al.*, 2007) pointed out that in the biquadratic spin-1 Heisenberg chain, see Eq. (27), the two-site entropy is ideal to highlight the presence of a dimerized phase. They also considered the two-site entropy for the ionic Hubbard model (Hubbard and Torrance, 1981).

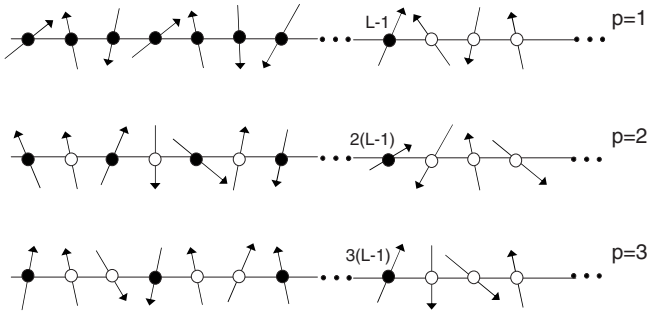

FIG. 8. The idea of the comb partition suggested by Keating Mezzadri, and Novaes illustrated for three different values of the spacing p. In all cases the subsystem A, denoted by solid circles contains l spins (be careful with the different notation with original paper) while the other (denoted by empty circles) contains the rest of the chain. The case $p=1$ corresponds to the well-known "block" division. From Keating *et al.*, 2006.

The idea of partitioning the system in a more elaborate way in order to analyze additional properties of entanglement lead to the introduction of the concept of comb entanglement (Keating *et al.*, 2006). This is illustrated in Fig. 8. The two blocks A and B are not chosen contiguous but A consists of ℓ equally spaced spins, such that the spacing between spins in this subsystem is p sites on the chain while B contains the remaining spins. For this choice of the partition, the "surface" separating the two blocks grows with the system size (different from the case $p=1$ where it is composed by two links). As a consequence nonlocal properties of entanglement between the two blocks can be investigated. For $p>1$ the leading contribution to the entropy scales linearly with block size $S_\ell(p)=\mathcal{E}_1(p)\ell+\mathcal{E}_2(p)\ln\ell$. In the limit $p\to\infty$ the coefficient $\mathcal{E}_1(p)$ is, to leading order, a sum from single spin contributions. The unexpected result is that the corrections vanish slowly, as $1/p$, different from other measures like concurrence where these long-range corrections are not present.

We conclude this section by discussing the single copy entanglement E_1 introduced by Eisert and Cramer (2005) and studied in detail for one-dimensional spin systems (Eisert and Cramer, 2005; Peschel and Zhao, 2005; Orus *et al.*, 2006). Single copy entanglement $S_{\mathrm{sc},\ell}$ quantifies the amount of entanglement that can be distilled from a single specimen of a quantum system. For spin chains it can be shown (Eisert and Cramer, 2005) that single copy entanglement asymptotically is half of the entanglement entropy $\lim_{L\to\infty} S_{\mathrm{sc},\ell}/S_\ell = 1/2$. This result was later generalized to conformally invariant models (Peschel and Zhao, 2005; Orus *et al.*, 2006).

2. *XY* chains and free fermion models

As shown in Sec. III.A.1.a the *XY* chain [$\Delta=0$ in Eq. (23)] can be mapped onto a model of free fermions. As a result an analytical (albeit nontrivial) approach is possible to calculate the block entropy. An analytic proof of the logarithmic dependence on the block entropy in the isotropic *XY* model was obtained by Jin and Korepin (2004b); Its *et al.* (2005); Franchini *et al.* (2006, 2007). The relation between the entanglement entropy of this model and the corner transfer matrices of the triangular Ising lattice has been derived by Peschel (2004). Keating and Mezzadri considered a more general free fermion Hamiltonian in which the matrices **A** and **B** [defined in Eq. (24)] do not have the tridiagonal structure which appear in the *XY* model (Keating and Mezzadri, 2005). They showed that under certain conditions the entropy can be expressed in terms of averages over ensembles of random matrices. In this section we recall the main steps of the derivation leading to the evaluation of the entanglement entropy, more details can be found in Latorre, Rico, and Vidal (Latorre *et al.*, 2004) and in the above-mentioned papers (Jin and Korepin, 2004b; Its *et al.*, 2005; Keating and Mezzadri, 2005; Franchini *et al.*, 2006, 2007).

The reduced density matrix of a block of ℓ spins can be expressed in terms of averages of strings of ℓ spin operators with weights given by the averages of these strings on the ground states. By mapping the spin operators in terms of the Majorana fermions, $a_{2l-1}=(\Pi_{m<l}\sigma_m^z)\sigma_l^x$, $a_{2l}=(\Pi_{m<l}\sigma_m^z)\sigma_l^y$, and given the fact that the resulting fermionic Hamiltonian is quadratic (Wick theorems holds), it is possible to express the block entropy in terms of the elements of the correlation matrix B_ℓ,

$$B_\ell=\begin{bmatrix} \Pi_0 & \Pi_1 & \cdots & \Pi_{\ell-1} \\ -\Pi_1 & \Pi_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ -\Pi_{\ell-1} & \cdots & \cdots & \Pi_0 \end{bmatrix}, \tag{45}$$

where $\Pi_l=\left[\begin{smallmatrix} 0 & g_l \\ -g_{-l} & 0 \end{smallmatrix}\right]$ with real coefficients g_l given as, for $L\to\infty$,

$$g_l=\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-il\phi}\frac{\cos\phi-\lambda-i\gamma\sin\phi}{|\cos\phi-\lambda-i\gamma\sin\phi|}. \tag{46}$$

It is crucial to note that the matrix B_ℓ are block Toeplitz matrices, that can be considered as usual Toeplitz matrices but with noncommuting entries. After the transformation of B_ℓ into a canonical form $\oplus_{m=1}^{L}=\nu_m i\sigma_y$, the system is described by a set of l independent two-level systems. Therefore the entanglement entropy is given by

$$S_\ell=-\sum_{m=1}^{\ell}\left[\frac{1+\nu_m}{2}\log\frac{1+\nu_m}{2}+\frac{1-\nu_m}{2}\log\frac{1-\nu_m}{2}\right]. \tag{47}$$

Numerical and analytical analysis of Eq. (47) has been performed leading to the behavior described in the preceding section. In order to obtain the analytical formula for the asymptotics of the entropy (47), the first step is to recast it, by the Cauchy formula, into a contour integral (Jin and Korepin, 2004b; Its *et al.*, 2005)

$$S=\lim_{\epsilon\to0^+}\lim_{\delta\to0^+}\frac{1}{2\pi i}\oint_{c(\epsilon,\delta)} e(1+\epsilon,\lambda)\frac{d\ln D_A(\lambda)}{d\lambda}d\lambda, \tag{48}$$

where $e(2x,2y)=-\Sigma_\pm(x\pm y)\log_2(x\pm y)$. The contour of integration $c(\epsilon,\delta)$ approaches the interval $[-1,1]$ as ϵ and δ tend to zero without enclosing the branch points of $e(1+\epsilon,\lambda)$. The matrix in $D_A(\lambda)=\det(\lambda I-B_\ell)$ is again of the block Toeplitz type. The asymptotics of the entropy can then be obtained from the asymptotics of $D_A(\lambda)$. This in turn can be done resorting to the Riemann-Hilbert approach to the theory of Fredholm integral equations (Bogoliubov *et al.*, 1993). This allows one to generalize the standard Szegö theorem for scalar Toeplitz matrices to obtain the leading formula for the determinant of the block Toeplitz matrix $D_A(\lambda)$. This leads to the asymptotics, $\ell\to\infty$, for the entropy

$$S=\frac{1}{2}\int_1^\infty \ln\left(\frac{\theta_3[\beta(x)+(\sigma\tau/2)]\theta_3[\beta(x)-(\sigma\tau/2)]}{\theta_3(\sigma\tau/2)}\right)dx, \tag{49}$$

where $\theta_3(s,\tau)$ is one of the Jacoby elliptic functions, $\beta(x)=(1/2\pi i)[\ln(x+1)-\ln(x-1)]$, $\sigma=1,0$ for $h_z<,>h_c$ and $\tau\sim\log(|1-\lambda^{-1}|)$. The critical behavior of Eq. (49) can be obtained by the asymptotic properties of $\theta_3(s,\tau)$ for small τ, and the leading term of the critical entropy results $S=-(1/6)\log[|1-\lambda^{-1}|/(4\gamma)^2]+\mathcal{O}[|1-\lambda^{-1}|(\log|1-\lambda^{-1}|)^2]$. The previous expression can be obtained in a more direct way via a duality relation connecting the quantum Ising chain with the Ising model on a square or triangular lattice (Peschel *et al.*, 1999; Calabrese and Cardy, 2004; Peschel, 2004). In particular the reduced density matrix can be written as the trace of the corresponding (corner) transfer matrix. In this way, however, the expression for the critical entropy can be obtained only for $\gamma^2+h_z^2>1$.

In the isotropic case $\gamma=0$, $c=1$ and then the prefactor of the log divergence is $1/3$. In this case the critical entropy can be obtained as an average, by realizing that the block Toeplitz matrices (for $\gamma=0$) are unitary. Then the contour integral can be recast into an integral in the ensemble of unitary matrices. This observation (Keating and Mezzadri, 2005) allows are to relate the spectral statistics of the model with the entanglement encoded in the ground state, following a reasoning that has many analogies in spirit with random matrix theory. The entanglement entropy was obtained explicitly for matrices **B**, in Eq. (24), being elements of classical groups. It is interesting that only **B** effects the prefactor of the logarithm in entropy, which is proportional to 2^{w_G}, where w_G is a universal quantity related to the classical group establishing the symmetries, and the constant of proportionality is Hamiltonian dependent.

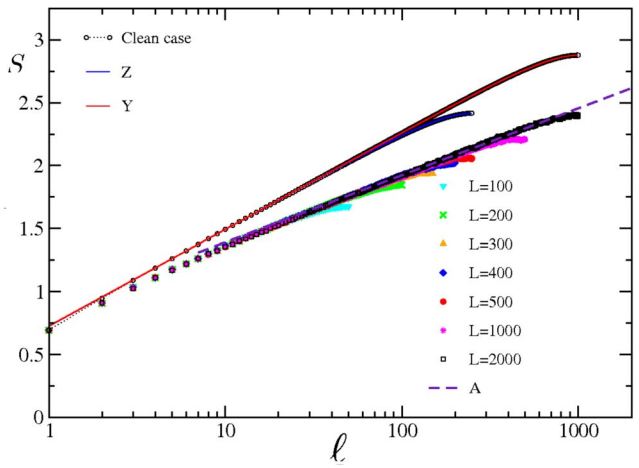

FIG. 9. (Color online) Entanglement entropy of a subsystem of size ℓ embedded in a closed ring of size L, shown vs ℓ in a log-linear plot. Numerical results obtained by exact diagonalization performed at the XX point. For clean nonrandom systems with $L=500$ and $L=2000$ (open circles), $\mathcal{S}_l$ is in agreement with Eq. (42) $A=0.8595+(\ln 2/3)\ln \ell$, $Y=0.72602+\frac{1}{3}\ln[(2000/\pi)\sin(\pi\ell/2000)]$, $Z=0.72567+\frac{1}{3}\ln[(500/\pi)\sin(\pi\ell/500)]$. From Laflorencie, 2005.

3. Disordered chains

Conformal invariance implies universal properties for the entanglement entropy. What happens when conformal invariance is lost as in the case of certain one-dimensional disordered spin systems? Refael and Moore (2004) were the first to look at this question by computing the block entropy for the Heisenberg, XX, and quantum Ising chains with random nearest-neighbor coupling. Their approach was based on a real space renormalization group developed earlier (Ma *et al.*, 1979; Fisher, 1994) for random spin chains where disorder is relevant and drives the system at low energies in the so-called random singlet phase which can be thought of as a collection of singlet bonds of arbitrary length. Consequently, the entropy of a given chain segment is $\ln 2$ times the number of singlets crossing the boundary between the two regions in which the systems is partitioned. Refael and Moore showed that the entropy, as in the case of clean critical chain, grows as the logarithm of block size $S_\ell \sim \tilde{c}\log \ell$ with a renormalized central charge $\tilde{c}=c\ln 2$.

A numerical test of this prediction was performed both for the XX (Laflorencie, 2005) and for the Heisenberg models (De Chiara *et al.*, 2005). In Fig. 9 we report the data of Laflorencie, the two curves represent the results for the clean and disordered case and fully confirm the prediction of Refael and Moore. The result that the ratio between the random and pure prefactor values of the block entropy is the same for the different chains studied by Refael and Moore (2004) suggest that this value might be determined by flow from the pure to the random fixed point. This conjecture was questioned by analyzing the entanglement entropy for a family of models which includes the N-states random Potts chain and the Z_N clock model. In this case it was shown that the ratio between the entanglement entropy in the pure and disordered systems is model dependent (Santachiara, 2006).

4. Boundary effects

Boundaries or impurities may alter in a significant way the the entanglement entropy. The result given in Eq. (42) was obtained for periodic boundary conditions. If the block is at the boundary of the chain then the prefactor is modified and the block entropy is one-half of the one given in Eq. (42) (Calabrese and Cardy, 2004):

$$S_\ell = \frac{c}{6}\log_2\left[\frac{2L}{\pi a}\sin\left(\frac{\pi}{L}\ell\right)\right] + g + \frac{A}{2}, \tag{50}$$

where A is the nonuniversal constant given in Eq. (42) and g is the boundary entropy (Affleck and Ludwig, 1991). The case of open boundary conditions in critical XXZ chains was also reconsidered by Laflorencie *et al.* (2006). In addition to the log divergence there is a parity effect depending on the number of spins of the block being even or odd. The amplitude of the resulting oscillating term decays as a power law with the distance from the boundary. The origin of this oscillating term is easy to understand qualitatively as an alternation of strong and weak bonds along the chain. The boundary spin has a strong tendency to form a singlet pair with its nearest neighbor on the right-hand side; due to the monogamy of the entanglement this last spin will be consequently less entangled with its partner on the third site of the chain. Furthermore, it was also shown that the alternating contribution to the entanglement entropy is proportional to a similar term in the energy density (the constant of proportionality related to the lattice constant and velocity of excitations). The effect of open boundary conditions on the entanglement entropy of a resonant valence bond solid was studied as well (Fan *et al.*, 2007). In this case, however, corrections due to open ends decay exponentially.

Different types of boundaries can appear in the AKLT quantum spin chain, with bulk spin 1 and two spin 1/2 at the ends. The entanglement entropy has been studied by Fan *et al.* (2004). They showed that the block entropy approaches a constant value exponentially fast with ℓ.

The entanglement entropy of one-dimensional systems is affected by the presence of impurities in the bulk (Levine, 2004; Peschel, 2005; Zhao *et al.*, 2006) or aperiodic couplings (Igloi *et al.*, 2007). In these cases the entanglement entropy has the same form as in Eq. (42) but with an effective value which depends on the strength of the defect. The entanglement properties of anisotropic open spin-one-half Heisenberg chains with a modified central bond were considered by Zhao *et al.* (2006) where the entanglement entropy between the two half chains was calculated using the DMRG approach. They find logarithmic behavior with an effective central charge varying with the length of the system. Numerical simulations of Zhao *et al.* (2006) showed that by going

from the antiferromagnetic to the ferromagnetic case the effective central charge grows from zero to one in agreement with Levine (2004). The combined presence of interaction between the excitation and a local impurity modifies the properties of a one-dimensional system. Starting from the work of Kane and Fisher (1992) it is now understood that at low energies scattering with the impurity is enhanced or suppressed depending on the interaction being repulsive or attractive. It is therefore expected that the entanglement entropy is affected as well. Levine, by means of bosonization, studied the entanglement entropy in a Luttinger liquid interrupted by an impurity and found that there is a correction to S due to the impurity which scales as $\delta S_{\mathrm{imp}} \sim -V_B \log(l/a)$, where V_B is the renormalized backscattering constant (Levine, 2004). In the repulsive case the backscattering flows to large values suggesting that the total entropy would vanish (the correction is negative). In the opposite case of attractive interactions, the impurity potential is shielded at large distances and the entropy would approach the value of the homogeneous liquid.

The single copy entanglement in the presence of boundaries has been considered as well (Zhou *et al.*, 2006). Different from the bulk contribution here the boundary contribution to the von Neumann entropy equals that of the single copy entanglement.

Some of these results provided fertile ground to study the entanglement encoded in the Kondo cloud. Specifically, the block entropy S_{imp} of a spin cloud with radius r around the impurity with the rest of the system has been analyzed (Sørensen *et al.*, 2007). By using a combination of Bethe ansatz results, conformal field theory, and DMRG methods, these authors demonstrated that S_{imp} is a universal scaling function of r/ξ_K.

B. Harmonic chains

Static systems of harmonic chains have been first analyzed by Audenaert *et al.* (2002), where periodic arrangements of harmonic oscillator modes have been considered. The oscillators have been coupled via their coordinate variables and the Hamiltonian has been chosen to be translational invariant. The entanglement in these systems has been analyzed for both the ground state and thermal states; both belong to the class of Gaussian states. Here we review the results obtained by Audenaert *et al.* (2002). For work on higher-dimensional lattices and emphasis on the entropy area law see Sec. V.C.2.

Using the covariance matrix defined in Eq. (35), the logarithmic negativity can be expressed in terms of the interaction matrix $\mathbb{U}$ (Audenaert *et al.*, 2002) $E_N = \mathrm{tr}\log_2 \mathbb{U}^{-1/2} P \mathbb{U}^{1/2} P$, where P is a diagonal matrix, with nonzero entries $P_{jj}=-1$ when the partial transposition is performed and $P_{jj}=1$ elsewhere. This entanglement monotone has been analyzed for bipartitions of a ring containing an even number of oscillators. It is convenient to define $\mathbb{U}=\left(\begin{smallmatrix} U' & U'' \\ U'' & U' \end{smallmatrix}\right)$. For the symmetric bisection into equally large connected parts, a lower bound for the logarithmic negativity has been obtained as

$$E_N \geqslant \frac{1}{2}\left|\mathrm{tr}\, F_n \log_2 \mathbb{U}\right| = \frac{1}{2}\log_2\left(1+4\sum_{m=0}\alpha_{2m+1}\right), \qquad (51)$$

where the coefficients α_n have been defined in Sec. III.D. F_n is the $n\times n$ flip matrix with 1 in the cross diagonal and 0 elsewhere. Equality holds if $F_{n/2}U''$ is semidefinite, which is the case for nearest-neighbor interaction. For this case one obtains $E_N^{\mathrm{NN}}=\frac{1}{2}\log_2(1+4\alpha_1)$. Remarkably, this result is independent of the ring size. This also tells us that the negativity of the symmetric bisection for a model including couplings α_d of arbitrary range is higher than that of the corresponding chain with only nearest-neighbor coupling and coupling strength $\Sigma_d\alpha_d$. It is interesting to note here that, for critical systems, the lowest eigenvalue of $\mathbb{U}$ tends to zero with growing system size. This leads to a symplectic eigenvalue of V that diverges with the system size with a consequent divergence of the negativity.

Analysis of general bisections revealed that for nearest-neighbor couplings the negativity of a single oscillator with the rest of the chain monotonically decreases with the size of the chain. This single-oscillator negativity turned out to establish a lower bound for the negativity of any connected set of two or more oscillators with the rest of the same chain (see top panel of Fig. 10). In all cases the maximum negativity has been observed for the symmetric bisection. Both features are expected to be generic to coupled ensembles of harmonic oscillators (Audenaert *et al.*, 2002). In particular, the infinite size limit of the symmetric bipartition negativity establishes an upper bound: $\lim_{m\to\infty} E_N(m,m) \geqslant E_N(n_1,n_2)$. This upper bound appears as a plateau in the top panel of Fig. 10, which as a function of n_1 and n_2 is reached for not too small n_1 and n_2. The plateau value is essentially proportional to the average energy per oscillator. With increasing nearest-neighbor coupling strength, a more shallow approaching of the plateau value is observed (see bottom panel of Fig. 10).

The situation changes when the negativity of two disconnected parts of the chain is considered. The particular limiting case of an alternating bipartition, consisting in all oscillators located at odd and even sites, has been analyzed in the presence of nearest-neighbor couplings only (Audenaert *et al.*, 2002). In this setting the logarithmic negativity is no longer limited by the average energy per oscillator but instead it grows linearly with system size n. The conclusion is that one part of the negativity can be related to the energy; the second part can be seen as a surface term, proportional to the area of the boundary forming the contact between both bipartitions. In one spatial dimension this "area" is the number of contacts between both parts, which in the periodic setting described above equals n. This interpretation finds further support in the result for the logarithmic negativity of a symmetric bisection in an open chain of oscillators, which is then roughly half the logarithmic negativity of the corresponding chain with periodic closure [see discussion in Audenaert *et al.* (2002)].

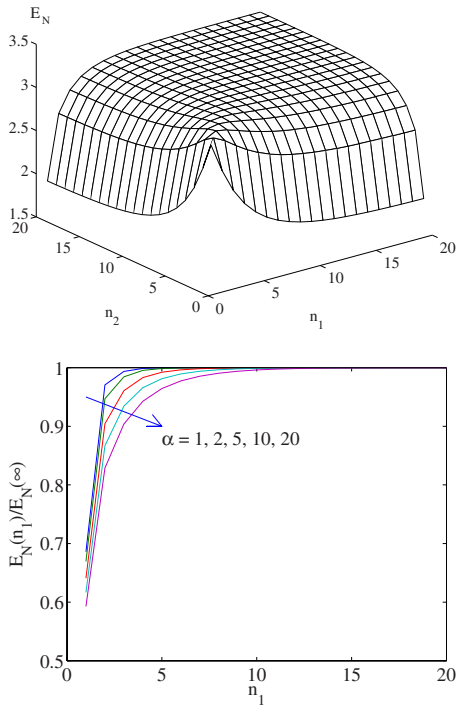

FIG. 10. (Color online) Logarithmic negativity for harmonic chains. Top panel: Logarithmic negativity for coupling strength $\alpha=20$ and a bipartition in n_1 and n_2 oscillators (total number of oscillators $N=n_1+n_2$). For sufficiently large parts, a plateau is reached. For one part consisting of only a few oscillators, the negativity decreases as a function of the system size. Bottom panel: The logarithmic negativity relative to its plateau value $E_N(\infty)$ for $n_2=20$ as a function of n_1 and varying coupling strength α. For small coupling the plateau limit is reached faster; since the plateau limit is connected to a local quantity (the average energy per oscillator), this can be explained by a longer correlation length that grows with α. From Audenaert *et al.*, 2002.

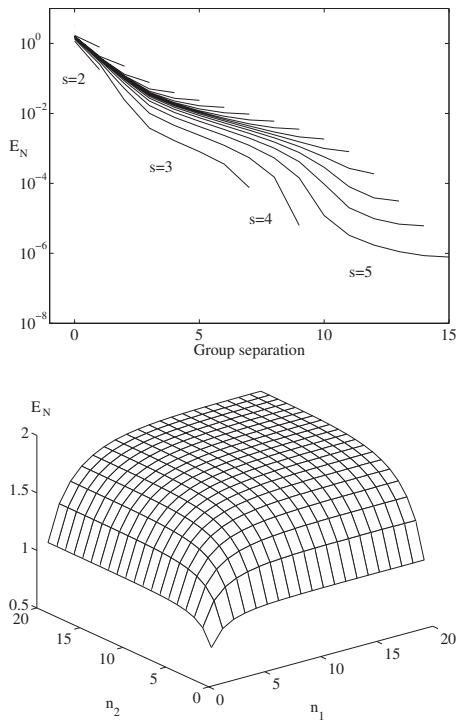

FIG. 11. Logarithmic negativity for harmonic chains. Top panel: The logarithmic negativity of two connected groups of s oscillators as a function of the distance between both groups. It is seen that the range of the negativity is finite but grows with the group size s. For $s=1$, its range is zero, meaning that no two single oscillators share pairwise entanglement unless they are neighbors. Bottom panel: Logarithmic negativity as in the top panel but for an open chain. The value of the negativity is roughly half as large as for the periodic chain since the number of connection points of the parts is halved. In contrast to the periodic chain, the negativity of one small part with the rest of the open chain grows with system size From Audenaert *et al.*, 2002.

An interesting puzzle results from analysis of the negativity of two connected chain parts that are separated from each other by a finite number of sites. Besides an expected all-over exponential decay of the logarithmic negativity with the distance between the two equally large groups, the negativity is also of limited range (see top panel of Fig. 11). This range increases with the size of the two parts. In particular, there is no pairwise negativity between two single oscillators unless they are neighbors. This implies that the entanglement of a distant group of oscillators cannot be due to "free" pairwise entanglement of single oscillators (as opposed to "bound" entanglement not detected by the negativity). So either "bound" pairwise entanglement is responsible for the entanglement present in distant groups or multipartite entanglement might play an important part.

We mention that pairwise correlations between single oscillators do exist notwithstanding a vanishing pairwise negativity (Audenaert *et al.*, 2002). In any case the presence of correlations is necessary for quantum entanglement of the parts. It is worth noting that both the plateau exhibited by the negativity for not too small size of the parts and the decrease of the single oscillators' negativity with the system size find a plausible explanation merely in terms of the correlation length. The same plausibility arguments predict the single oscillators' negativity to increase with growing system size when open chains are considered; in fact this is what these authors observed [see Audenaert *et al.* (2002)]. The observed short range of the negativity in particular for small connected sets of oscillators overstretches this simple reasoning and demonstrates that the connection between entanglement and correlations is indeed more subtle.

C. Systems in $d>1$ and the validity of the area law

The scaling of the entanglement entropy in systems of higher dimensions has been the subject of intense investigation in various fields of research. In the context of quantum information the understanding of the entropy scaling as a function of block size has important consequences on the simulability of a quantum system by a

classical algorithm and therefore is attracting a lot of interest. The picture that emerged from the analysis of the one-dimensional case, i.e., the violation of the area law when the system is critical, does not seem to hold in higher dimensions. The situation appears more complex. The higher dimensional generalization of matrix product states, the projected entangled pair states, satisfy the area law (Verstaete and Cirac, 2004; Verstaete *et al.*, 2006) despite a divergent correlation length. Another example is the ground state of antiferromagnetic Ising-spin networks embedded on planar cubic lattices (Wellard and Orus, 2004) where the area law is fulfilled at the critical point. In the following we concentrate on the ground state of some many-body Hamiltonian for which the block entropy has been recently computed. We first consider the case of hopping Hamiltonian of (free) fermions and bosons and then discuss the case of harmonic lattices (note that historically the bosonic case was considered first).

1. Fermi systems

In one dimension, by virtue of the Jordan-Wigner transformation, the block entropy of a system of interacting spins is tightly connected to that of a (free) Fermi gas. It is of great interest to understand what are the properties of the block entropy for free fermions in d dimensions. This has been studied by Gioev and Klich (2006) and Wolf (2006) where it was shown that logarithmic corrections persist in higher dimensions,

$$S_\ell \sim \ell^{d-1} \log_2 \ell . \tag{52}$$

The expression for the constant of proportionality in the equation above has been obtained by Gioev and Klich (2006) resorting to the Widom conjecture. Wolf (2006) exploited the quadratic lower bound of Fannes *et al.* (2003). The corrections to the area law are a Fermi surface effect. In the case of fractal dimension of either the Fermi or the block surface, the scaling is modified to $S_\ell^{fr} \sim \ell^{d-\beta} \log_2 \ell$, where $1-\beta$ is the maximum fractal enhancement of dimension of either the Fermi or the block boundary. An interesting case where the system undergoes a Lifhitz phase transition has been considered by Cramer *et al.* (2007). As it was pointed out by Cramer *et al.* these transitions related to a change in the Fermi surface manifest in nonanalytic behavior of the prefactor of the leading order term entanglement entropy.

For regular block and Fermi surfaces, numerical analysis has confirmed the modified area law for critical two-dimensional (Barthel, Chung, and Schollwöck, 2006; Li *et al.*, 2006) and three-dimensional (Li *et al.*, 2006) models. Barthel, Chung, and Schollwöck (2006) studied the tight binding model as a two-dimensional model with a connected Fermi surface as well as the model $H=-\Sigma_{x,y}\{[1+(-1)^y]c_{x,y}^\dagger c_{x,y+1}+c_{x,y}^\dagger c_{x+1,y+1}+c_{x,y}^\dagger c_{x-1,y+1}\}$ with a disconnected Fermi surface and $H=-\Sigma_{x,y}\{hc_{x,y}^\dagger c_{x+1,y}+[1+(-1)^{x+y}]c_{x,y}^\dagger c_{x,y+1}\}$ with a zero-dimensional Fermi surface, for the one-dimensional case. Whereas in the first two cases, the entropy is found to obey the modified area law, this is no longer true for the third model with zero-dimensional Fermi surface. There corrections to the standard area law $S\sim\ell^{d-1}$ are sublogarithmic.

The same feature has been observed by Li *et al.* (2006) studying the spinless fermionic in two and three spatial dimensions. These authors conjectured an interesting connection between the modified area law and the density of states at the Fermi energy. They formulated this in terms of the *codimension* at the Fermi energy: i.e., the dimension of momentum space minus the dimension of the degeneracy at the Fermi energy in momentum space, providing a measure of the relative portion of gapped excitations in the low lying spectrum of the model. In agreement with the findings for the model zero-dimensional Fermi surface, these authors observed only sublogarithmic corrections to the area law if the codimension at the Fermi energy is 2. From this they conjectured that in two spatial dimensions a codimension less or equal to 1 is necessary for the modified area law $S\sim\ell\log_2\ell$ to apply. They do not mention implications of fractal codimension due to a fractal Fermi or block surface. This finding would be worth further investigation to higher dimensions in order to fix the connection between area law and codimension at the degeneracy point. We finally mention the interesting connection between the block entropy and the Berry phase in lattice models of fermions discussed by Ryu and Hatsugai (2006).

2. Harmonic systems

Harmonic systems have also been investigated to understand the validity of the area law. They provide one of the few physical systems for which exact analytical treatements are avaliable [see Cramer *et al.* (2006), and references therein].

We first consider a system of noncritical harmonic oscillators with nearest-neighbor interaction and periodic boundary conditions. Noncriticality implies that the lowest eigenvalue of the interaction matrix $\mathbb{U}$, $\lambda_{\min}(\mathbb{U})$, is well separated from zero. Further peculiar characteristics of the covariance matrix, in particular its symmetric and circulant form, allow one to give estimates for upper and lower bounds of the block entropy of some compact d-dimensional hypercubic region with edge length ℓ and surface proportional to ℓ^{d-1} (Plenio *et al.*, 2005). The upper bound has been established directly from the logarithmic negativity [see Audenaert *et al.* (2002)], whereas for the lower bound several estimates for the dominant eigenvalue of the reduced density matrix have been employed.

Key ingredients to the problem are the largest eigenvalue of the covariance matrix and the uncertainty relation which constrains all eigenvalues of the covariance matrix to lie above $\hbar/2$. The result is that both bounds are proportional to ℓ^{d-1}, hence the entanglement entropy is proportional to the surface of the block. An extension to general block shapes has been formulated as well for Gaussian states (Cramer *et al.*, 2006). For pure states, lower and upper bounds are given as before,

which both scale linearly with the surface area of the block. For a given finite range of interaction beyond nearest neighbors, the area law can be stated only in terms of an upper bound [as in Plenio *et al.* (2005), this upper bound has been obtained from the logarithmic negativity]. Therefore Cramer *et al.* (2006) could not exclude the block entropy to scale with lower dimensionality than the block area. The area law can even be present in disordered systems; the requirement for the area law to hold in this case is that $\kappa := \lambda_{\max}(\mathbb{U})/\lambda_{\min}(\mathbb{U})$ is bounded for all admissible disorder realizations. The presence of a finite temperature T enters only in the proportionality factor, when thermal equilibrium states (which are still Gaussian) are considered.

It is interesting to note that an algebraically diverging correlation length does not automatically imply a violation of the area law. Cramer *et al.* (2006) found that for noncritical systems of harmonic oscillators and algebraic decay $r^{-\alpha}$ of the correlations with distance r still leads to the area law as long as $\alpha > 2d$. Essential for this conclusion is that κ is bounded. These authors conjectured d as a tighter bound at least for hypercubic blocks (when compared to the one-dimensional Ising model, already at $\alpha = d = 1$ logarithmic corrections appear).

For critical systems the situation is different: criticality leads to eigenvalues of $\mathbb{U}$ arbitrarily close to zero and hence unbound κ. Unanyan and Fleischhauer (2005) reported a one-to-one connection between criticality and logarithmic corrections to the area law for one-dimensional systems with finite range interaction. These authors also reported evidence for this connection to hold in higher dimensional setups. However, the absence of logarithmic corrections for critical two-dimensional arrays of harmonic oscillators and nearest-neighbor interaction has been evidenced numerically in Barthel, Chung, and Schollwöck *et al.* (2006a). The analytical calculations of Cramer *et al.* (2007) concluded the area law to apply for arbitrary number of dimensions, opposed to the conclusions of Unanyan and Fleischhauer (2005). As support for their claim, Unanyan and Fleischhauer (2005) quoted a factoring interaction matrix, which, however, corresponds to a noninteracting array of one-dimensional harmonic chains; this can be seen as the limiting case of an anisotropic interaction with finite range and does not give support to the original claim. It is indeed fundamentally different from harmonic d-dimensional lattices with isotropic finite range interaction, as discussed by Barthel, Chung, and Schollwöck (2006) and Cramer *et al.* (2007). Nevertheless it raises the question for a critical anisotropy for the coupling of the harmonic oscillators, which on the background of the findings by Barthel, Chung, and Schollwöck *et al.* (2006) and Cramer *et al.* (2007) could be phrased as: "Does a finite critical anisotropy exist beyond which the harmonic lattice is quasi one-dimensional?"

Logarithmic corrections are also observed, when infinite range interactions are considered, which drive the system towards criticality (Unanyan *et al.*, 2007). To this end, these authors considered a two-dimensional array of harmonic oscillators, with an interaction of finite range in the x direction and an infinite range in the y direction. This is an instructive example in that it leads to the logarithmic correction $\sim l_x \ln l_y$. Interestingly, the logarithm contains the length l_y of the block, where the interaction has infinite range. The prefactor of the logarithm is half the length of the block in the x direction (with finite range interaction).

D. LMG model

The logarithmic divergence with the block size of the entanglement entropy is not exclusive of one-dimensional systems. The block entropy of the LMG model was studied in both the ferromagnetic (Stockton *et al.*, 2003; Latorre, Orus, *et al.*, 2005; Barthel, Dunsuel, and Vidal, 2006b; Vidal *et al.*, 2007) and antiferromagnetic (Unanyan *et al.*, 2005) cases. In the LMG model each spin is interacting with all other spins in the network; therefore, the idea of a block as depicted in Fig. 6 does not fit very well. Nevertheless, it is perfectly legitimate to define the reduced entropy of ℓ spins once the other $N-\ell$ (N is the total number of spins) have been traced out. Evidently the entropy is independent on which spins have been selected to be part of the block. In Fig. 12 the representative behavior of the entanglement entropy as a function of the various regions of the phase diagram is shown for a fixed value of the block and system sizes. Below we summarize the main findings related to the LMG model. In the ferromagnetic case and in the case of $\gamma = 1$ (isotropic mode), the entropy diverges logarithmically with the block size $S \sim (1/2)\log \ell$ while at fixed l and N diverges when the external magnetic field approaches 1 from below. Also in the antiferromagnetic, case (Unanyan *et al.*, 2005), when no transition as a function of the field is present, the entropy grows logarithmically with the size of the block in the isotropic limit. Different from the one-dimensional case where the prefactor is universal and related to the central charge, here the origin of the prefactor of the logarithmic divergence is related to the presence of the Golstone modes and to the number of vanishing gaps. Vidal *et al.* clarified this issue by studying a number of collective spin models by means of $1/N$ expansion and scaling analysis (Vidal *et al.*, 2007).

E. Spin-boson systems

The entropy in models of spins interacting with harmonic oscillators have been analyzed as well. Here the separation between spin and bosonic degrees of freedom is natural and the partition leads to study the reduced entropy of one subsystem (say the spin). The entanglement entropy was studied for the Jaynes-Cummings (Bose *et al.*, 2001), Tavis-Cummings model (Lambert *et al.*, 2004, 2005), and for the spin-boson model (Costi and McKenzie, 2003; Jordan and Buttiker, 2004; Stauber and Guinea, 2006a; Kopp *et al.*, 2007). Lambert and coworkers analyzed how the super-radiant quantum phase transition manifests in the entanglement between the

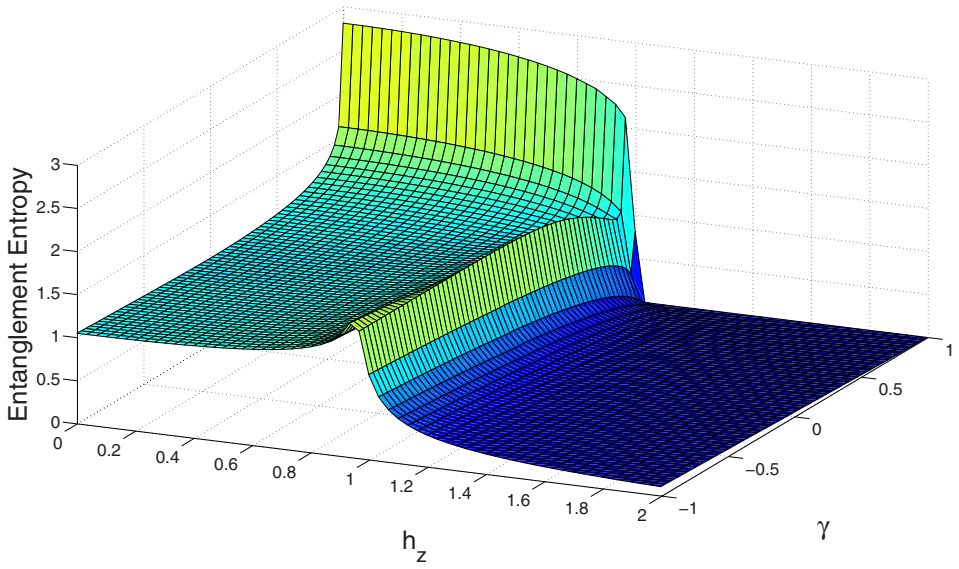

FIG. 12. (Color online) Entanglement entropy for $N=500$ and $\ell=125$ as a function of the parameters h_z and γ of the ferromagnetic LMG model. For $\gamma \neq 1$, there is an anomaly at the critical point $h_z=1$, whereas the entropy goes to zero at large h_z since the ground state is a fully polarized state in the field direction. In the zero field limit, the entropy saturates when the size of the system increases. For $\gamma=1$, the entropy increases with the size of the system in the region $0 \leqslant h_z < 1$ and jumps to zero at $h_z=1$. From Latorre, Orus, *et al.*, 2005.

atomic ensemble and the field mode. They computed the von Neumann entropy numerically at finite N and analytically in the thermodynamical limit. They found that the entropy diverges at the phase transition as (see Fig. 13)

$$S \sim -(1/4)\log|\lambda - \lambda_c|, \tag{53}$$

where λ is the coupling between the spins and the boson field and λ_c is the value at which the transition takes place (see Fig. 13). The entropy of the spin-boson model [see Eq. (32)] was studied by numerical renormalization group by Costi and McKenzie (2003). An analytic calculation, including other dissipative models, has been presented by Stauber and Guinea (2004) and more recently by Stauber and Guinea (2006a, 2006b) and Kopp *et al.* (2006). The broken-symmetry state has an effective classical description and the corresponding von Neumann entropy is zero. In the symmetric phase the entropy can be easily expressed as a function of the ground state energy and δ [defined in Eq. (32)]. At the transition point the entropy is discontinuous with a jump given by Kopp *et al.* (2007) as $\Delta S = \ln 2 + \delta/4\delta_c \ln(\delta/\delta_c)$ (δ_c is a high energy cutoff). A systematic analysis of the entropy in the spin-boson model for different coupling regimes was pursued recently by Kopp and Le Hur (2007) and Le Hur *et al.* (2007).

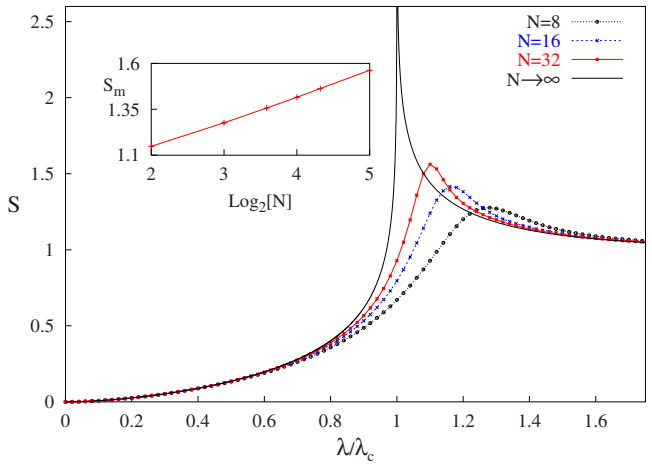

FIG. 13. (Color online) Entropy S between two-level atom and cavity field as a function of the atom-radiation coupling for both $N \to \infty$ and finite N. Inset: The scaling of the value of the entanglement maximum as a function of $\log N$. From Lambert *et al.*, 2004.

We finally mention the interesting connection between entanglement and energy fluctuations introduced by Jordan and Buttiker (2004) and exploited in detail both for a spin and for harmonic oscillators coupled to a bath. This connection might be useful in light of possible experimental measure of entanglement [see also Klich *et al.* (2006)]. For example, as pointed by Jordan and Buttiker (2004), in certain mesoscopic realization of qubits as metallic rings of superconducting nanocircuits, a measurement of persistent current can be directly related to a measurement of the entropy.

F. Local entropy in Hubbard-type models

An important class of interacting fermion models is that of Hubbard type models (see Sec. III.B). First studies of entanglement in the one-dimensional case have appeared in Gu *et al.* (2004) and in Korepin (2004). Most of the studies in this type of systems analyzed the properties of the local entropy.

Gu *et al.* analyzed the local entropy for the one-dimensional extended Hubbard model for fermions with spin 1/2. Due to the conservation of particle number and z projection of the spin, the local density matrix of the system takes the simple form

$$\rho_j^{(1)} = z|0\rangle\langle 0| + u^+|\uparrow\rangle\langle\uparrow| + u^-|\downarrow\rangle\langle\downarrow| + w|\uparrow\downarrow\rangle\langle\uparrow\downarrow|,$$

independent of the site number j because of translational symmetry. The broken translation invariance in the charge density wave phase has not been taken into

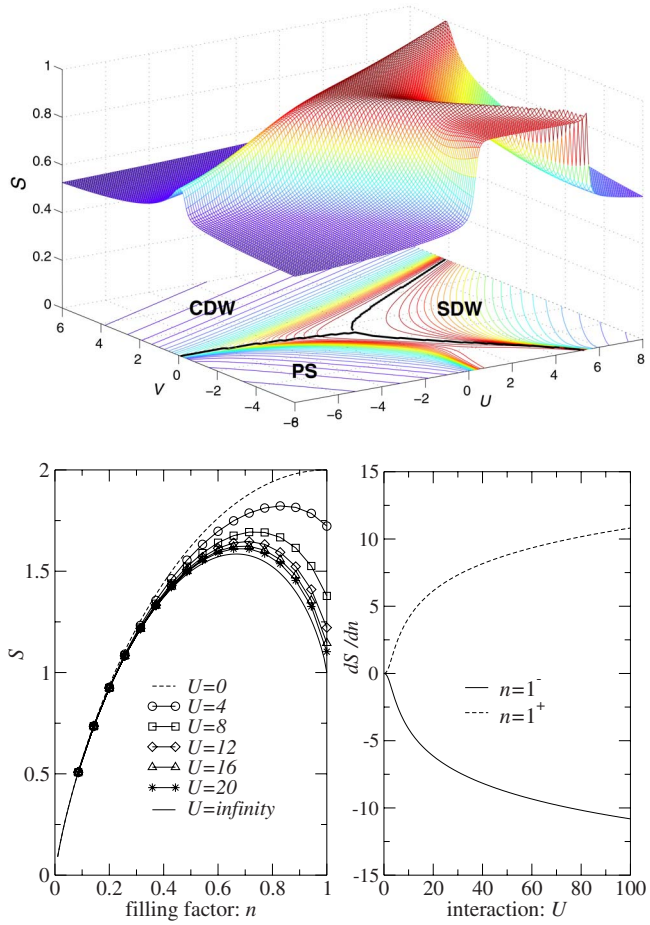

FIG. 14. (Color online) Local entanglement entropy in Hubbard models. Upper panel: The local entropy as a function of the on-site coupling u and nearest-neighbor coupling v. The contour plot below includes the known phase diagram of the model (full solid line). Except the superconducting phase, the phase diagram is nicely reproduced. Lower panel: The local entropy for the standard Hubbard model and different on-site couplings u as a function of the filling (1 corresponds to half filling). Except for $U/t=0$, where a maximum appears, this quantity shows a dip at half filling, i.e., at the metal-insulator transition. From Gu *et al.*, 2004.

account in this work. This does not affect the central result but might affect the entropy within the charge density wave phase. Except the superconducting phase, the phase diagram at half filling (for $\mu=0$) of this model has been nicely reproduced by the contour plot of the local entropy (see top panel of Fig. 14), where the phase transition coincides with its crest. This turns out to be a general feature of local entropies—also for spin models—as opposed to entanglement class specific measures, as, e.g., the concurrence for pairwise entanglement whose maxima in general appear at a certain distance to quantum critical points and hence are not associated to the quantum phase transition. In view of the monogamy of entanglement this is interpreted as evidence of dominant multipartite entanglement in the vicinity of quantum phase transitions.

For the Hubbard model Eq. (30) and fixed U/t, the local entropy as a function of n shows a dip at the critical filling fraction $n_c=1$, where a metal-insulator transition occurs (for $U>0$) (see bottom panel of Fig. 14). For the two limiting cases $U=0,\infty$ the maximum instead is located at filling fractions, where the ground state is a singlet of the largest symmetry group. Gu *et al.* conjectured that this was true for general $U>0$ and then the presence of an unknown phase transition at these maxima.

This analysis points out that the local entropy indicates different phase transitions in different ways, essentially depending on whether the quantity is sensitive to its order parameter or not. Due to the u(1) symmetry of the model, the single site reduced density matrix is a function of occupation numbers only. These operators cannot, however, describe order parameters of superconductivity or some order parameter of the metal-insulator transition. Indeed, the superconducting phase can be predicted if the entropy of entanglement is calculated for a block of spins, instead of for just a single site (Deng *et al.*, 2006). A reduced density matrix of at least two sites is necessary for superconducting correlations [see also Legeza and Sólyom (2006) for a similar result obtained for the ionic Hubbard model] to be seen.

Another model studied is the so-called bond-charge extended Hubbard model (see Sec. III.B). In phases II and III of Fig. 15 there are superconducting correlations which are due to η pairing and hence indicate the presence of multipartite entanglement, as discussed before (Vedral, 2004a). At the bond-charge coupling corresponding to $x=1$, the entanglement of the model has been analyzed by Anfossi, Giorda, *et al.* (2005) and for general x and $n=1$ by Anfossi, Degli Esposti Boschi, *et al.* (2006) and Anfossi *et al.* (2007). Besides the local entropy of entanglement S_i, Anfossi, Giorda, *et al.* (2005) used the negativity (Vidal and Werner, 2002) and the quantum mutual information (Groisman *et al.*, 2005). While S_i measures all (pairwise and multipartite) quantum correlations involving this specific site, the negativity offers a lower bound for the quantum correlation of two specific sites, and mutual information accounts for pairwise quantum and classical correlations. Therefore this combination of correlation measures opens the possibility to decide what type of correlation is relevant at a quantum phase transition. The results are shown in the uper panel of Fig. 15. The different phases are shown in Fig. 15: they are discriminated by local occupation numbers as described in the top panel; consequently, the entropy S_i bears the information on all phase diagrams except the insulating line IV. This is seen from the plot of $\partial_u S_i$ (with $u \doteq U/t$) as a function of the on-site Coulomb coupling u and the filling fraction n. A comparison of first derivatives with respect to $y=n,u$ (depending on the phase transition) of all three correlation measures reveals common singularities for $\partial_y S_i$ and $\partial_y \mathcal{I}_{ij}$ only for the transitions II-III and II-IV; furthermore, for both it was proven that the range of the concurrence R diverges (Anfossi, Degli Esposti Boschi, *et al.*, 2006). These facts allow one to characterize the transitions II-III and II-IV (at $n=1$ and arbitrary x) as governed by pairwise en-

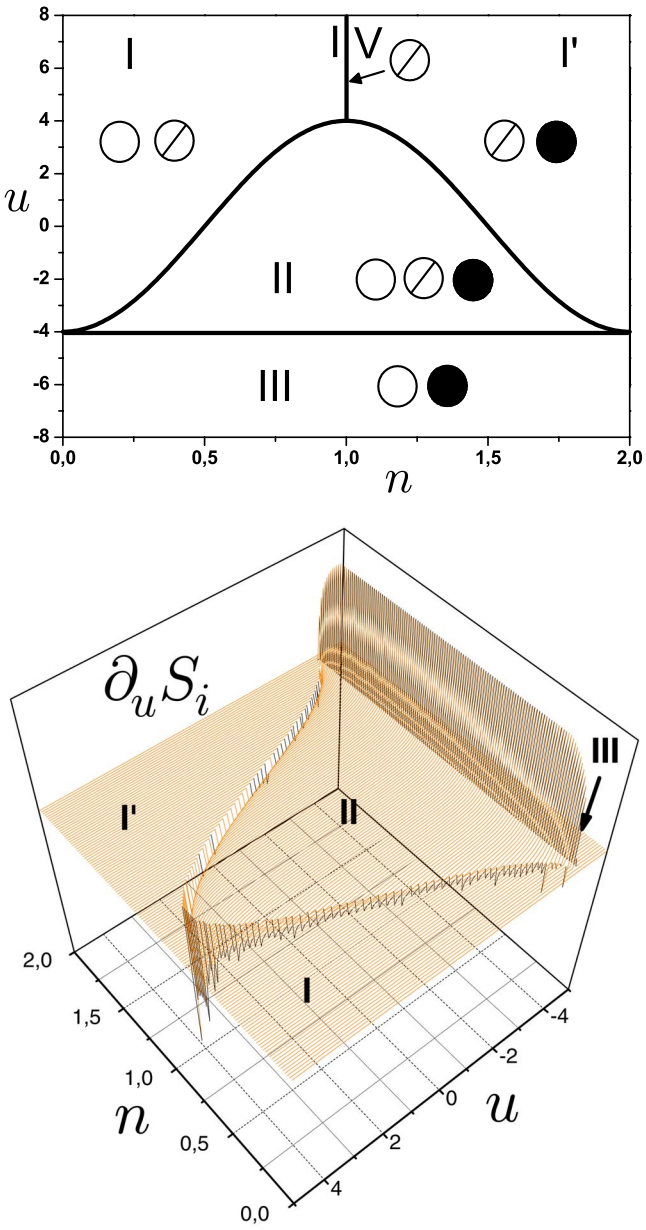

FIG. 15. (Color online) Local entanglement entropy in bond-charge Hubbard models. Upper panel: The ground state phase diagram of the Hirsch model at $x=1$. Empty, slashed, and solid circles indicate the presence of empty, singly, and doubly occupied sites, respectively. Bottom panel: Except the insulating line IV, the phase diagram is nicely reproduced by $\partial_u S_i$. From Anfossi, Degli Esposti Boschi, *et al.*, 2006.

tanglement, that is the more spread along the chain the closer the transition is gotten. For the transitions II-I and II-I′ instead the multipartite entanglement is relevant, with a finite range of the concurrence. Similar behavior was encountered for *noncritical* spin models where the divergence of R is accompanied by the emergence of a fully factorized ground state (see Secs. III and IV.A.1). Here R diverges close to QPT; it was also noted by Anfossi *et al.* that, while the ground state in IV is indeed factorized, phase III is made of a superposition of doubly occupied and empty states. Observing that the pairwise entanglement is vanishing in both phases III and IV, these authors conjectured that the divergence of R marks an "entanglement transition" solely in the pairwise entanglement.

In order to detect the transition II-IV at $n=1$ and any x, $\partial_x S_i$ has been calculated by means of DMRG (Anfossi, Degli Esposti Boschi, *et al.*, 2006). Its singularities allow one to accurately determine the charge gap as a function of the bond-charge coupling x.

We now proceed with the Hubbard model in a magnetic field (see Sec. III.B) Also here, the local entropy S_i has been looked at in order to analyze its entanglement. As in the examples before, S_i indicates the second order phase transitions in terms of divergences of its derivatives $\partial_h S_i$ and $\partial_\mu S_i$, respectively. Indeed, it has been demonstrated that $\partial_h S_i$ and $\partial_\mu S_i$ can be expressed in terms of spin and charge susceptibilities (Larsson and Johannesson, 2005), hence bridging explicitly the gap between the standard method in condensed matter physics for studying phase transitions and the approach from quantum information theory.

The local entropy for the bosonic version of the Hubbard model, the Bose-Hubbard model, was computed by Giorda and Zanardi (2004) and by Buonsante and Vezzani (2007) for different graph topologies. The focus of this work was on the dependence of the entanglement on the hopping amplitude. These authors showed that for certain classes of graphs the local entropy is a nonmonotonic function of the hopping. Also for the bosonic case the local entropy is a good indicator for the presence of a (superfluid-insulator) quantum phase transition.

Summarizing, the body of work developed suggests that local entropies can detect QPTs in systems of itinerant fermions, particularly if the transition itself is well predicted by a mean field approach for local observables of the model [see also Larsson and Johannesson (2006)]. Furthermore, translational invariance is necessary for the prediction to be independent of the site, which the local entropy is calculated for. If this symmetry is absent, it might prove useful to average over the sites; the resulting measure is then equivalent to the Q measure (Meyer and Wallach, 2002).

G. Topological entanglement entropy

We close this section by summarizing the ongoing research studying subleading corrections to the block entropy in the two-dimensional systems. Most of the results were demonstrated for quantum two-dimensional lattices (though generalizations to higher dimension is straightforward).

Fradkin and Moore considered quantum critical points in two spatial dimensions with scale invariant ground state wave functions, characterizing, for example, the scaling limit of quantum eight vertex models and non-Abelian gauge theories [see Ardonne *et al.* (2004)]. The main result is that a universal logarithmically divergent correction, determined by the geometry

of the system's partition, emerges in addition to the area's law term in the entanglement block entropy (Fradkin and Moore, 2006).

Such work benefits of the earlier seminal contribution of Kitaev and Preskill (2006) and Levin and Wen (2006) demonstrating that the correction to the area law is of topological origin. Namely, the entanglement entropy was demonstrated to scale as

$$S = \alpha\ell - \gamma + O(\ell^{-1}). \tag{54}$$

The coefficient α is nonuniversal and ultraviolet divergent. In contrast, the quantity γ can be expressed as $\log D$, where D is known as the total quantum dimension, it is universal, and related to the topological winding number of the theory. Calculations were pursued with methods of topological field theory, giving an explicit expression for γ.

We remark that the result of Kitaev and Preskill and Levin and Wen provides an alternative avenue to lattice gauge theory methods (Wen, 2004), detecting a genuine topological order in the system (when descriptions based on local order parameters fail) by direct inspection of the wave function.

When a topological order is present, the ground state of the system acquires a peculiar degeneracy when its lattice has a nontrivial genus g. Hamma *et al.* studied the entropy of the Kitaev model on a two-dimensional lattice with generic genus g. The Kitaev model (Kitaev, 2003) is a two-dimensional exactly solvable lattice model with double periodic boundary conditions, whose Hamiltonian can be realized with a set of spins in a square lattice with ring exchange and vertex interactions. The ground state of the Kitaev model is characterized by the presence of a topological order.[6] For such systems Hamma *et al.* related the degeneracy of the ground state (that is, 4^g) to the block entropy (Hamma *et al.*, 2005a, 2005b).

As a step toward models with more generic topological orders, the topological entropy was studied numerically for the quantum dimer model in a triangular lattice (Furukawa and Misguich, 2007) and for fractional quantum Hall states (Haque *et al.*, 2007). We finally mention that the topological term in the entanglement entropy in the context of quantum gravity was evidenced by Fursaev (2006) and Ryu and Takayanagi (2006b).

H. Entanglement along renormalization group flow

One of the original motivations put forward by Preskill to investigate entanglement in many-body systems (Preskill, 2000) was the idea that quantum information could elucidate some features of the renormalization group which is a cornerstone method in modern physics. It is natural to think that the procedure of tracing out high energy modes in a renormalization group step has some kind of irreversibility built in. Quantum information concepts could prove to be useful in elucidating issues related to this irreversibility and possible help could come from relating the celebrated c theorem (Zamolodchikov, 1986) to the loss of information.

Entanglement loss along a renormalization group trajectory was studied in spin chains (Latorre, Lütken, *et al.*, 2005). Recently a number of relations relating renormalization group, conformal field invariance, and entanglement loss were derived by Orus (2005). According to Latorre, Lütken, *et al.* (2005) entanglement loss can be characterized at three different levels.

(1) Global entanglement loss. By using the block entropy as a measure of entanglement, for which we know the result of Eq. (42), and an inequality on the central charges which derives from the c theorem, it follows that $S^{\mathrm{UV}} \geqslant S^{\mathrm{IR}}$. The block entropy at the ultraviolet fixed point cannot be smaller than that at the infrared fixed point.

(2) Monotonous entanglement loss. It is also possible to follow the entanglement along the entire transformation. In this case the entropy is a nondecreasing function along the flow. As a simple example one can consider the block entropy of an Ising chain close to the critical point which goes as $S \sim \ln|\lambda - 1|$, from which monotonicity follows.

(3) Fine-grained entanglement loss. The monotonicity of the entanglement appears present at a deeper level in the structure of the density matrix. One can show (Latorre, Lütken, *et al.*, 2005; Orus *et al.*, 2006) through majorization relations that the spectrum of the reduced density matrix gets more ordered along the flow. By denoting r_m the eigenvalues of the reduced density matrix ρ, majorization relation between the two sets of spectra (corresponding to two different parameters) means that there is a set of relations for which $\Sigma_{i=1,n} r_i \geqslant \Sigma_{i=1,n} r_i'$ for $n = 1, \ldots, d$ (d is the dimension of ρ).

Motivated by ultraviolet divergencies of the entropy of entanglement in quantum field theory, Casini and Huerta introduced a quantity $F(A,B)$ related to the entropy measuring the degree of entanglement between two regions A and B. The function F is defined as $F(A,B) = S(A) + S(B) - S(A \cap B) - S(A \cup B)$ which coincides with the mutual information Eq. (9) in the case of nonintersecting regions. In two dimensions, it is a finite positive function with the property $F(A,B) \leqslant F(A,C)$ for $B \subset C$ if $A \cap C = \varnothing$. Then, for sets with a single (path-connected) component in two-dimensional conformal field theories they showed that it allows one to prove an alternative entropic version of the c theorem (Casini and Huerta, 2007).

VI. LOCALIZABLE ENTANGLEMENT

A. Localizable entanglement and quantum criticality

The study of localizable entanglement (see Sec. II.C) in spin chains allows one to find a tighter connection

[6] The Kitaev model was suggested to provide a realization of the so-called "toric code," namely, a topological quantum computer made by a physical system with anionic excitations (see also Castagnoli and Rasetti, 1993).

between the scales over which entanglement and correlations decay (as shown previously the two spin entanglement, expressed by the concurrence, does not decay on the same range of correlations) (Verstraete, Martin-Delgado, and Cirac, 2004; Popp *et al.*, 2005, 2006). One expects that the procedure of entangling distant sites by a set of local measurements will be less effective as the distance the two particles increases thus leading to a definition of entanglement length ξ_E. For a translational invariant system ξ_E can be defined in analogy of the standard correlation length:

$$\xi_E^{-1} = -\lim_{|i-j|\to\infty} \log \frac{E_{\mathrm{loc}}(|i-j|)}{|i-j|}. \tag{55}$$

By definition the entanglement length cannot be smaller than the correlation length $\xi_E \geqslant \xi$, therefore at a second order phase transition the localizable entanglement length diverges. In addition, there may also appear "transition points" associated solely to a divergence in ξ_E. In order to avoid misinterpretations, it must be stressed that the *localizable* "classical" two-point correlations then diverge as well. Thus the essence of the phenomenon is that correlations can be localized between arbitrarily distant sites by means of suitable local operations and classical communication despite a finite correlation length; necessary for this is the presence of global entanglement (Popescu and Rohrlich, 1992).

For the Ising model in a transverse field it can be shown that (Verstraete, Popp, and Cirac, 2004)

$$\max_{\alpha=x,y,z} |Q_\alpha^{ij}| \leqslant E_{\mathrm{loc}}(i-j) \leqslant \frac{1}{2}\sum_{\pm} \sqrt{s_\pm^{ij}}, \tag{56}$$

where $s_\pm^{ij} = (1 \pm \langle S_i^z S_j^z\rangle)^2 - (\langle S_i^z\rangle \pm \langle S_j^z\rangle)^2$ and $Q_\alpha^{ij} = \langle S_i^\alpha S_j^\alpha\rangle - \langle S_i^\alpha\rangle\langle S_j^\alpha\rangle$. In this case, the lower bound in Eq. (56) is determined by the two-point correlation function in the x direction. In the disordered phase ($\lambda < 1$) the ground state possesses a small degree of entanglement and consequently its entanglement length is finite. The situation changes at the other side of the critical point. Here, although the correlation length is finite, the entanglement length is infinite as asymptotically the correlation tends to finite values. The divergence of ξ_E indicates that the ground state is a globally entangled state, supporting the general idea that multipartite entanglement is most relevant at the critical point (Osborne and Nielsen, 2002; Roscilde *et al.*, 2004).

The properties of localizable entanglement were further investigated for a spin-1/2 XXZ chain by Jin and Korepin (2004a) and Popp *et al.* (2005) as a function of the anisotropy parameter Δ and of an externally applied magnetic field h. These authors used exact results for correlation functions relying on the integrability of the models to find the bounds in Eq. (56). For the antiferromagnetic XXX case they provided the lower bound $E_{\mathrm{loc}}(i-j) \geqslant (2/\pi^{2/3})\ln|i-j|/|i-j|$. The presence of the anisotropy increases the lower bound of the localizable entanglement. At the Berezinskii-Kosterlitz-Thouless critical point ($\Delta = 1$) the lower bound of the nearest-neighbor localizable entanglement shows a kink (Popp *et al.*, 2005). As pointed out this might have implications in the general understanding of the Berezinskii-Kosterlitz-Thouless phase transitions where the ground state energy and its derivatives are continuous as well as the concurrence (see Sec. IV.A.2 and Fig. 3).

The localizable entanglement in the two-dimensional XXZ model was discussed as well (Syljuåsen, 2003b) by means of quantum Monte Carlo simulations. A lower bound has been determined by studying the maximum correlation function which for $\Delta > -1$ is Q_x, the long-range (power-law) decay of the correlation implying a long-ranged localizable entanglement.

The definition of localizable entanglement has an interesting connection with the concept of quantum repeaters introduced by Briegel *et al.* (1998). Quantum repeaters have been designed to enhance the transmission of entanglement through noisy channels. The idea is to distribute along the channels a number of intermediate sites where a certain number of local operations are allowed in order to maximize the entanglement between the transmitter and a receiver. This is the very definition of localizable entanglement.

Localizable entanglement has been defined in Sec. II.C as an average over all possible measuring processes, it is of interest to also understand the statistical fluctuations around this average value. To this end Popp *et al.* (2005) analyzed the variance associated to the entanglement fluctuations: $\delta E_{\mathrm{loc}}^2 = \Sigma_s p_s E^2(|\psi_s\rangle) - E_{\mathrm{loc}}^2$, where E is a measure of pairwise entanglement. Fluctuations of the entanglement increase in the vicinity of a critical point. This was checked explicitly for the one-dimensional Ising model. As detailed below, additional results were obtained for spin-1 systems where a true transition in the entanglement (with a diverging ξ_E but finite correlation length) has been found.

B. Localizable entanglement in valence bond ground states

For half-integer spins, gapped nondegenerate ground states are characteristic for systems in a disordered phase (consider paramagnets, for example). A finite gap in the excitation spectrum of the system in the thermodynamic limit makes the correlations decaying exponentially. This is the Lieb-Schultz-Mattis theorem, establishing that, under general hypothesis, the ground state of a spin system is either unique and gapless or gapped and degenerate (Lieb *et al.*, 1961) [see Hastings (2004)]. It was a surprise when Haldane discovered that systems of integer spins can violate this theorem (Haldane, 1983a, 1983b). This suggests to investigate whether the entanglement in the ground state might play some role in establishing the hidden order characteristic for the Haldane phases. An aspect that might be relevant to this aim was addressed by studying the localizable entanglement in AKLT models (Verstraete, Martin-Delgado, and Cirac, 2004). The ground state of this class of models is the valence bond type as discussed in Sec. III.A.4:

$$|gs\rangle_{\mathrm{AKLT}} = (\otimes_k A_{k,\bar{k}})|I\rangle_{\bar{01}}|I\rangle_{\bar{12}}\cdots|I\rangle_{\bar{N}N+1}. \quad (57)$$

$|I\rangle$ are singlets and A are 3×4 operators projecting the Hilbert space of two combined spins on its symmetric part, at the given site. This is nothing other than a matrix product state (see Sec. III.A.4). For this state it was demonstrated that a singlet state made of two spins $1/2$ located at the ends of the chain can be always found (see Fig. 1). This implies that the localizable entanglement is long ranged despite the exponentially decaying correlation (Verstraete, Martin-Delgado, and Cirac, 2004).

The localizable entanglement can be related to the string order parameter $O^{\alpha}_{\mathrm{string}}$ defined in Eq. (28). The key to this relation is to observe that the localizable entanglement can be calculated as an expectation value (Campos Venuti and Roncaglia, 2005) as $L(|\psi\rangle) = \langle\psi|\sigma_0 G_s(\psi)\sigma_N|\psi\rangle$, where $G_s = \Sigma_s |s\rangle\langle\bar{s}|\mathrm{sgn}(\langle\psi|\sigma^y\otimes\sigma^y|\psi\rangle)$, and $|s\rangle$ is the optimal basis which maximizes the entanglement of assistance. For the AKLT model, the expression above reads

$$L(|gs\rangle_{\mathrm{AKLT}}) = \left\langle \prod_{i=1}^{N} e^{i\pi S_i^y} \right\rangle. \quad (58)$$

In this case, both the localizable entanglement and all three components of the string order parameter saturate. Perturbing the AKLT ground state, namely making the resonating valence bonds with nonmaximally entangled states $(|10\rangle - e^{-i2\phi}|01\rangle)/\sqrt{2}$, the relation between the hidden order and the localizable entanglement is weaken (as compared to the AKLT model): It has been demonstrated that the string order parameters for $\alpha = x,z$ are finite, while the localizable entanglement vanishes exponentially with the deformation ϕ (Popp *et al.*, 2005), but a tight connection of the localizable entanglement and O^{y}_{string} is observed to persist also for the ϕ deformed ground state (Campos Venuti and Roncaglia, 2005).

The valence-bond-solid phase order was further studied by looking at the hidden order in chains with more complicated topology. The von Neumann entropy was studied in the spin-1 XXZ model with biquadratic interaction and single ion anisotropy by Wang *et al.* (2005); Gu *et al.* (2006); and by Campos Venuti, Degli Esposti Boschi (2006). Some features of the corresponding phase diagram are captured. The Haldane transitions exhibited in the phase diagrams are marked by anomalies in the von Neumann entropy; its maximum at the isotropic point is not related to any critical phenomenon (the system is gapped around such a point), but it is due to the equiprobability of the three spin-1 states occurring at that point (Campos Venuti, Degli Esposti Boschi, *et al.*, 2006). Since the Berezinskii-Kosterlitz-Thouless transition separating the XY from the Haldane or large-D phases connects a gapless with a gapped regime, it was speculated that an anomaly in the entanglement should highlight such a transition (Gu *et al.*, 2006).

VII. THERMAL ENTANGLEMENT

Though the very nature of entanglement is purely quantum mechanical, we saw that it can persist for macroscopic systems and will survive even in the thermodynamical limit. Entanglement survives also at finite temperature. This temperature could be as high as 100 K in high-temperature superconductors (Vedral, 2004a; see also Fan and Lloyd, 2005). In this section we review the properties of entanglement in many-body systems at finite temperatures (see also Anders and Vedral, 2007). We show that the analysis could shed new light on the interplay between the quantum nature of the system and its thermodynamics. Moreover, somewhat surprisingly, macroscopic state variables can be used to detect entanglement. Thermodynamics describes large scale systems with macroscopic properties, state variables T,N,V,p, external fields h, and response functions of the susceptibility and heat capacity, respectively, χ, C, etc. Addressing entanglement as a thermodynamical property raised a significant amount of interest in various communities. One wants to know, for example, under what conditions can we detect and extract entanglement. Can we see entanglement itself as a state variable, just like pressure is for a collection of atoms in a gas? What could be the corresponding thermodynamical potential? Is entanglement extensive? Since entanglement is closely related to entropy, we would expect the answer to the last question to be "yes."

The states describing a system in thermal equilibrium states are determined by the Hamiltonian and the inverse temperature $\beta = (1/T)$. The density matrix is $\rho = Z^{-1}e^{-\beta\hat{H}}$, where $Z = \mathrm{tr}(e^{-\beta\hat{H}})$ is the partition function of the system. The thermal states expanded in the energy eigenbasis $|e_i\rangle, i = 0,1,\ldots$ are then

$$\rho = \frac{e^{-\beta E_0}}{\sum_i e^{-\beta E_i}}|e_0\rangle\langle e_0| + \frac{e^{-\beta E_1}}{\sum_i e^{-\beta E_i}}|e_1\rangle\langle e_1| + \cdots. \quad (59)$$

Any separable state, or classical state, with respect to this split into subsystems $A,B,C,D,\ldots$ (for example, the sites of a spin system) can then be written as a convex mixture of tensor products of states of the respective subsystems $A,B,C,D,\ldots$ with probabilities p_i, $\rho = \Sigma_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C \otimes \rho_i^D \otimes \cdots$. If the state in Eq. (59) cannot be written in the form given above then it is entangled. In this section we discuss the properties of this *thermal entanglement*.

A. Thermal pairwise entanglement

Extensive efforts have been made to quantify thermal entanglement in many-body systems starting from Arnesen *et al.* (2001); Gunlycke *et al.* (2001); and Nielsen (2001). Several models of interacting spins in arrays were discussed. Entanglement as measured by concurrence was shown to exist at nonzero temperatures in the transverse Ising (Osborne and Nielsen, 2002), Heisenberg (Wang, 2002b; Wang and Zanardi, 2002; Asoudeh and

Karimipour, 2004; Tribedi and Bose, 2006), *XXZ* (Rossignoli and Canosa, 2005; Canosa and Rossignoli, 2006), and *XYZ* models (Rigolin, 2004; Zhang and Zhu, 2006), ferrimagnetic transition (Wang and Wang, 2006), and spin-one chains (Zhang and Li, 2005). Sevaral nontrivial aspects of the behavior of the pairwise entanglement at finite temperatures are illustrated by considering the simple case of two-site systems.

(*a*) *XXX* model. We start our discussion on thermal entanglement by considering the *XXX* antiferromagnet [see Eq. (23)]. In this case the thermal state of this system can be written as a Boltzmann mixture of the singlet and triplet states:

$$\rho_T = \frac{1}{Z} e^{3\beta J} |\psi^-\rangle\langle\psi^-| + e^{-\beta(J+2h)} |00\rangle\langle 00| + e^{-\beta(J-2h)} |11\rangle\langle 11| + e^{-\beta J} |\psi^+\rangle\langle\psi^+|, \tag{60}$$

where $Z = e^{3\beta J} + e^{-\beta(J+2h)} + e^{-\beta(J-2h)} + e^{-\beta J}$ is the partition function of the system and $|\psi^\pm\rangle = |01\rangle \pm |10\rangle$. For the sake of simplicity, we focus our attention on only two regimes.

The first regime is when the coupling J is large compared to the external field h. The ground state is then the singlet state, and at low temperature the system is therefore entangled. At higher temperatures the triplet becomes mixed into the singlet, and when (roughly) $T > J/k$ the entanglement completely disappears (when the external field is zero). Therefore in order to have high-temperature entanglement in dimers we need a large value of the coupling constant J.

When J is fixed, the second regime occurs when we vary the value of the external field h. When h is large (greater than $2J$), the ground state is $|11\rangle$ and at zero temperature the dimers are therefore not entangled. The point where the singlet state becomes replaced by $|11\rangle$ as the ground state corresponds to the quantum phase transition (occurring in the thermodynamical limit). However, if we start to increase the temperature, the singlet state—which is the first excited state under these circumstances—starts to become populated. Entanglement can then be generated by increasing the temperature. The behavior of entanglement as a function of the magnetic field is shown in Fig. 16.

(*b*) *Ising model*. Another interesting case is that of Ising coupling. At zero temperature only the lowest energy level is populated. In the case $N=2$ the tangle of this pure state can be calculated from the density matrix, for $h>0$, $\tau_1 = J^2/(J^2+h_z^2)$. It is clear that the entanglement is highest for nearly vanishing magnetic fields and decreases with increasing field amplitude (this expression, however, is not valid for strictly $h=0$, where no entanglement is present). We now turn our attention to the case of nonzero temperatures. For a general pure state only one of the eigenvalues of the Hamiltonian weight Eq. (59) is nonzero and therefore equal to the tangle. For low temperature and magnetic field, i.e., $h, T \ll J$, it is a good approximation to assume that only the two lowest energy levels are populated. In this case,

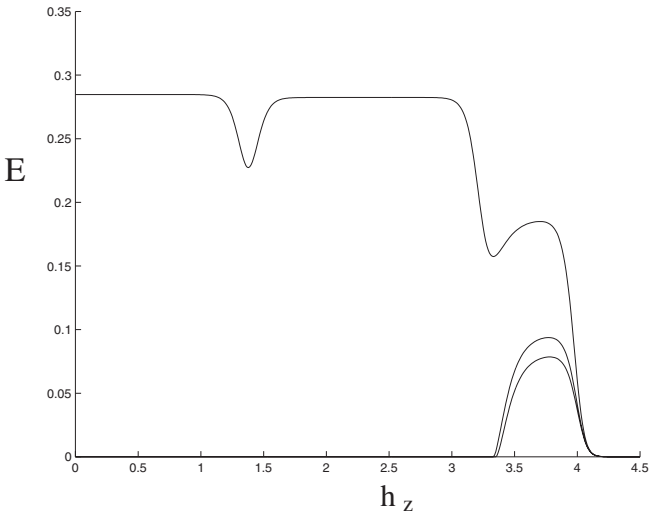

FIG. 16. Entanglement between two qubits interacting according to the antiferromagnetic Heisenberg model as a function of the external field h_z and temperature (multiplied by the Boltzmann's constant) T with coupling $J=1$. The topmost plot shows the variation of nearest-neighbor entanglement with the magnetic field. The middle and bottommost plots show the same for next-nearest and next-to-next-nearest neighbors, respectively, from Arnesen *et al.*, 2001.

the combination of the two lowest states also combines their concurrences in the following way (Gunlycke *et al.*, 2001):

$$C = \max\{|w_0 C_0 - w_1 C_1|, 0\}, \tag{61}$$

where the index 0 refers to the ground state, while 1 refers to the excited state, and w_0 and w_1 are the thermal weights of the ground and excited states, respectively; see Eq. (59). This phenomenon has been named as concurrence mixing. In this case, the first excited state is the Bell state $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and Eq. (61) reduces to $C = |w_0(J/\sqrt{J^2+h_z^2}) - w_1|$. In general, the first term in the above equation is larger than the second, and in this case the concurrence decreases with temperature as w_0 decreases and w_1 increases. Moreover, it is also possible to see that, for a given temperature, the entanglement can be increased by adjusting the magnetic field and is generally largest for some intermediate value of the magnetic field. This effect can be understood by noting that w_0 increases with increasing h as the energy separation between the levels increases, but $J/\sqrt{J^2+h_z^2}$ decreases. As a result the combined function reaches a peak as we vary h and decreases subsequently, inducing analogous behavior for the concurrence.

B. Pairwise entanglement in the $T \neq 0$ critical region

At finite temperatures but close to quantum critical points, quantum fluctuations are essential to describe the properties of the systems (Chakravarty *et al.*, 1989; Sachdev, 1999). In order to study the interplay between the thermal entanglement and quantum fluctuations

caused by the critical point at $T=0$ analysis on small clusters is no longer sufficient. For the presentation we consider the one-dimensional quantum XY models. Such systems cannot exhibit any phase transitions at finite temperature, but the existence of the quantum critical point is reflected in the crossover behavior at $T\neq 0$. The renormalized classical crosses over the quantum disordered regimes through the so-called quantum critical region (Sachdev, 1999). In the T-h plane a V-shaped phase diagram emerges, characterized by the crossover temperature customarily defined as $T_{\text{cross}} \doteq |\lambda^{-1}-\lambda_c^{-1}|$. For $T \ll T_{\text{cross}}$ the thermal de Broglie length is much smaller than the average spacing of excitations; therefore the correlation functions factorize in two contributions coming from quantum and thermal fluctuations separately. The quantum critical region is characterized by $T \gg T_{\text{cross}}$. Here we are in the first regime and the correlation functions do not factorize. In this regime the interplay between quantum and thermal effects is the dominant phenomenon affecting the physical behavior of the system.

Thermal entanglement close to the critical point of the quantum XY models was studied by Amico and Patané (2007). In analogy with the zero-temperature case they demonstrated that the entanglement sensitivity to thermal and to quantum fluctuations obeys universal $T \neq 0$ scaling laws. The crossover to the quantum disordered and renormalized classical regimes in the entanglement has been analyzed by the derivatives of the concurrence $\partial_\lambda C$ and $\partial_T C$. The thermal entanglement results to be very rigid when the quantum critical regime is accessed from the renormalized classical and quantum disordered regions of the phase diagram; such a "stiffness" is reflected in a maximum in $\partial_T C$ at $T \sim T_{\text{cross}}$. The maximum in the derivatives of the concurrence seems a general feature of the entanglement in the crossover regime. In this respect we mention also that the concurrence between two Kondo impurity spins discussed by Stauber and Guinea (2004, 2006b) experiences the largest variation in the crossover phenomenon.

Due to the vanishing of the gap at the quantum critical point, in the region $T \gg T_{\text{cross}}$ an arbitrarily small temperature is immediately effective in the system (see Fig. 17). From the analysis of the quantum mutual information [see Eq. (9)] it emerges that the contribution of classical correlations is negligible in the crossover, thus providing indication that such a phenomenon is driven solely by the thermal entanglement. It is interesting to study how the existence of the factorizing field h_f affects the thermal pairwise entanglement (vanishing at zero temperature). It is found that the two tangle τ_2 still vanishes in a region of the h-T plane, fanning out from just h_f (white region in Fig. 18) (Amico *et al.*, 2006). If entanglement is present it is shared between three or more parties.

It is further observed that, in contrast to the analysis of the ground state, at finite temperature we cannot characterize the two separate phases of parallel and antiparallel entanglement. In fact, the two types of entanglement (though well defined also for mixed states) can swap by varying T and/or r. The exchange between parallel and antiparallel entanglement occurs in a nontrivial way that ultimately produces the reentrance of τ_2 seen in Fig. 18.

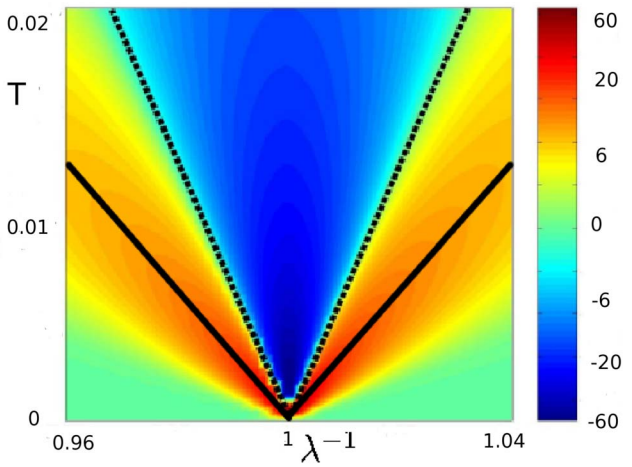

FIG. 17. (Color online) Temperature effect on the anomalies from the critical divergence of the field derivative $C(R)$ measured by $\partial_T[\partial_a C(R)]$. The density plot corresponds to $\gamma=1$ and $R=1$. $T=T^*$ and $T=T_M$ are drawn as dashed and thick lines, respectively. Maxima below T^* are found at $T_M=\beta T_{\text{cross}}$ with $\beta \sim 0.290 \pm 0.005$ and are independent of γ and R; the crossover behavior is enclosed between two flexes of $\partial_T[\partial_a C(R)]$ at T_{c1} T_{c2}; such values are fixed to $T_{c1}=(0.170\pm0.005)T_{\text{cross}}$ and $T_{c2}=(0.442\pm0.005)T_{\text{cross}}$ and independent of γ and R. For $T \lesssim T_{c1}$, $\partial_T[\partial_a C(R)] \simeq 0$. Scaling properties are inherited in $\partial_T[\partial_a C(R)]$ from $\partial_a C(R)]$. From Amico and Patané, 2007.

The common feature in all cases for which the existence of entanglement could be proven is that both high temperatures and high values of magnetic field move the thermal states away from the region with nonzero entanglement. This is understandable because high values of magnetic field tend to order all spins parallel to the field which corresponds to an overall state being a product of the individual spin states. There is upper limit of

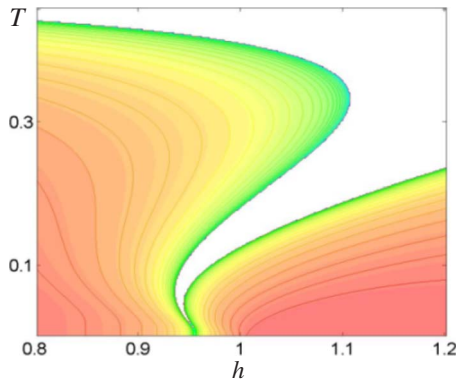

FIG. 18. (Color online) Contour plot of τ_2 in the h-T plane, for $\gamma=0.3$ (i.e., $h_f=0.9539\ldots$). The white area indicates where $\tau_2=0$. From Amico *et al.*, 2006.

this phenomenon, since an increase in the temperature is disruptive for entanglement due to thermal fluctuations.

C. Thermal entanglement witnesses

At first sight it may be surprising that thermodynamical variables can witness entanglement since what is required to obtain them are the system's eigenenergies, and no eigenstate information is required. Since entanglement resides in the fact that the state is inseparable (and is not related to the value of its energy), it would appear that the partition function is not enough to characterize entanglement. This logic, although seemingly simple, is not entirely correct. The reason is subtle and lies in the fact that the whole Hamiltonian is used for constructing the partition function, so in a roundabout way we do have information about the states as well. Entanglement witnesses in spin systems have been considered by Brukner and Vedral (2004); Tóth (2005); Wu *et al.* (2005); Hide *et al.* (2007) (see also the next section on multipartite entanglement).

We now illustrate how and why we can derive entanglement witnesses for the partition function. Suppose that the system is described by an antiferromagnetic Heisenberg model. First, we have the following identity coming from the Hellman-Feynman theorem $\Sigma_{i=1}^{N}\langle S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z\rangle \sim \frac{d}{dJ}(\ln Z)$. This means that the two point correlations function summed over all nearest neighbors can be derived from the partition function. This is also a quantity that can be accessed experimentally as is usually performed in solid state experiments. Most importantly, this average has in general different values for separable and entangled states. It can, therefore, serve as an entanglement witness as will be seen shortly. The fact that will emerge is that, in order to say if a state is entangled, we do not need to have the analytically calculated form of the eigenstates in order to tell if the resulting mixture is entangled. One price to pay for this is that we will only be able to derive a sufficient condition for entanglement that is typical of entanglement witnesses. Namely, we will be able to tell if for some conditions the resulting thermal state is entangled, but we will not be able to say with certainty that the state is not entangled if these conditions are violated.

Using $U=\langle H\rangle$ and $M^z=\Sigma_{j=1}^{N}\langle S_j^z\rangle$ we obtain

$$\frac{U+hM^z}{NJ} = -\frac{1}{N}\sum_{i=1}^{N}(\langle S_i^x S_{i+1}^x\rangle + \langle S_i^y S_{i+1}^y\rangle + \langle S_i^z S_{i+1}^z\rangle). \tag{62}$$

The right-hand side of Eq. (62) is an entanglement witness as shown by Tóth (2005): for any separable state, that is, for any classical mixture of the products states, $\rho=\Sigma_k w_k \rho_k^1\otimes\rho_k^2\otimes\cdots\otimes\rho_k^N$, one has

$$\frac{1}{N}\left|\sum_{i=1}^{N}(\langle S_i^x S_{i+1}^x\rangle + \langle S_i^y S_{i+1}^y\rangle + \langle S_i^z S_{i+1}^z\rangle)\right| \leqslant \frac{1}{4}. \tag{63}$$

This is also valid for any convex sum of product states (separable states). The upper bound was found using the Cauchy-Schwarz inequality and knowing that for any state $\langle S^x\rangle^2+\langle S^y\rangle^2+\langle S^z\rangle^2 \leqslant 1/4$. It is important to note that the same proof can also be applied if one considers the XX model. We now give our thermodynamical entanglement witness: if, in the isotropic XXX or XX Heisenberg model, one has

$$|U+hM^z| > N|J|/4, \tag{64}$$

then the solid state system is in an entangled state. The entanglement witness is physically equivalent to the exchange interaction energy or, equivalently, to the difference between the total (internal) energy U and the magnetic energy $-hM$. From the tracelessness of the Pauli operators one can see that $\lim_{T\to\infty} U\to 0$. This means that the value of the internal energy as given by Eq. (64) should be defined relative to the referent value of zero energy in the limit of high temperatures. In order to complete the proof we need to give an explicit example of a state that violates Eq. (63) (or the corresponding inequality for the XX model). This implies that Eq. (64) is indeed an entanglement witness and not just a bound that is trivially satisfied by any quantum state.

As an example of such a state we take the ground state of the antiferromagnetic isotropic XXX Heisenberg model with zero magnetic field. The energy of this state was found to be (Hulthén, 1938) $|E_0/JN| = 0.443\,25 > 1/4$. Furthermore, due to the symmetry of the XXX Heisenberg Hamiltonian one has $E_0/3NJ = \langle S_i^x S_{i+1}^x\rangle_0 = \langle S_i^y S_{i+1}^y\rangle_0 = \langle S_i^z S_{i+1}^z\rangle_0 = -1.773/12$ for every i. This implies that $(1/N)|\Sigma_{i=1}^{N}(\langle S_i^x S_{i+1}^x\rangle_0 + \langle S_i^y S_{i+1}^y\rangle_0)| = 0.295 > 1/4$. Therefore Eq. (64) is an entanglement witness for the solid state systems described by XXX or XX Heisenberg interaction.

We now discuss various spin interaction models some of which are exactly solvable and whose dependence of internal energy U and magnetization M on temperature T and magnetic field h are known. This will help us to determine the parameter regions of T and h where one has entanglement in the solids.

We first consider the XXX Heisenberg model with no magnetic field, in this case the magnetization vanishes and the thermodynamical witness, Eq. (64), reduces to $|U|>N|J|/4$. It was shown that concurrence $C(1)$ is zero at any temperature in the ferromagnetic case and that it is given by $C=\frac{1}{2}\max[0,|U|/(NJ)-1/4]$ in the antiferromagnetic case (Wang and Zanardi, 2002). Thus C is nonzero if and only if $|U|/(NJ)>1/4$. This shows that the thermodynamical entanglement witness can detect entire bipartite entanglement as measured by concurrence. Furthermore, the fact that the value of the entanglement witness for the ground state is well above the limit of $1/4$ suggests that entanglement could exist and be detected by the thermodynamical witness at nonzero tempera-

tures. In the presence of a finite magnetic field the low temperature partition function is given by $Z=e^{\beta(J+h_z)/4}(1+e^{-\beta h_z/2}N/\sqrt{2\pi\beta J})$. Using this we obtain $|U+h_zM|/(NJ)=1/4$ and thus no entanglement can be detected in agreement with Asoudeh and Karimipour (2004) and Pratt (2004).

The XX Heisenberg model with nonzero magnetic field is the most interesting as it is exactly solvable, the partition function was found by Katsura (1962). We introduce the dimensionless quantities $b=h_z/T$ and $K=J/T$ [note a difference of a factor of 2 in the definitions of J and K with respect to Katsura (1962)] and the function $f(K,b,\omega)=\sqrt{2K^2+2K^2\cos 2\omega-4bK\cos\omega+C^2}$ for convenience. Then the internal energy is given by (Katsura, 1962)

$$\frac{U}{N}=-\frac{T}{4\pi}\int_0^\pi f(K,b,\omega)\tanh f(K,b,\omega)d\omega, \tag{65}$$

and the magnetization by (Katsuram, 1962)

$$\frac{M}{N}=-\frac{1}{2\pi}\int_0^\pi \frac{4K^2\cos^2\omega}{f(K,b,\omega)}\tanh f(K,b,\omega)d\omega, \tag{66}$$

in both the ferromagnetic and antiferromagnetic cases.

We use Eqs. (65) and (66) to determine the parameter regions of temperature T and magnetic field h_z for which entanglement exists in the solid state system. The critical values of T and h below which entanglement can be detected is of the order of J, which can be as high as 10 K (Hammar *et al.*, 1999).

It should be stressed that the analysis based on the entanglement witness can be applied to any model for which we can successfully obtain the partition function. This feature is the main advantage of using the thermodynamic witnesses approach to detecting entanglement. This method for determining entanglement in solids within the models of Heisenberg interaction is useful in the cases where other methods fail due to incomplete knowledge of the system. This is the case when only the eigenvalues but not eigenstates of the Hamiltonian are known (which is the most usual case in solid state physics) and thus no measure of entanglement can be computed. Furthermore, in the cases where we lack the complete description of the systems one can approach the problem experimentally and determine the value of the thermodynamical entanglement witness by performing appropriate measurements. It is important to emphasize that any other thermodynamical function of state could be a suitable witness, such as the magnetic susceptibility or heat capacity (Wiesniak *et al.*, 2005); see next section.

The temperature as well as other thermodynamical state variables have been shown to behave also as entanglement witnesses of *spatial entanglement* (Anders *et al.*, 2006). This general feature was explicitly worked out in the case of a noninteracting bosonic gas. It was found that entanglement can exist at arbitrarily high temperatures, provided that one can probe smaller and smaller regions of space.

The methods outlined here are not only applicable to the models we considered. There are several interesting questions and possibilities for generalizations such as consideration of Hamiltonians with higher spins, two- and three-dimensional systems, non-nearest interactions, anisotropies, other thermodynamical properties (e.g., heat capacity, magnetic susceptibility), and so on. Similar analysis can be done for continuous thermal entanglement in a field. It has been shown that for noninteracting bosons entanglement exists when their de Broglie thermal wavelength is smaller than their average separation a (Anders *et al.*, 2006). The precise condition is $kT<\hbar^2/2ma^2$, where m is the mass of bosonic particles. We now introduce the following correspondence between spin coupling J and the continuous variables bosonic kinetic energy $J=\hbar^2/2ma^2$. This implies that we can think of the thermal de Broglie wavelength for spins as $\lambda_{dB}=a\sqrt{J/T}$, where a is the spin separation. The condition for entanglement that the wavelength is larger than the lattice spacing a now leads us to the condition that $T<J$ which is exactly the result obtained from a more detailed analysis above.

D. Experimental results

The question of having macroscopic entanglement is not only fascinating in its own right but it also has a fundamental significance as it pushes the realm of quantum physics well into the macroscopic world, thus opening the possibility to test quantum theory versus alternative theories well beyond the scales on which theirs predictions coincide. It also has important practical implications for implementation of quantum information processing. If the future quantum computer is to reach the stage of wide commercial application, it should share the same feature as the current (classical) information technology and be based on solid state systems. It will thus be important to derive the critical values of physical parameters (e.g., the high-temperature limit) above which one cannot harness quantum entanglement in solids as a resource for quantum information processing.

It was demonstrated experimentally that entanglement can affect macroscopic properties of solids, albeit at very low (critical) temperature (below 1 K) (Ghosh *et al.*, 2003). This result opens a possibility that purely quantum correlations between microscopic constituents of the solid may be detected by only a small number of macroscopic thermodynamical properties.

Ghosh *et al.* made measurement on an insulating salt, $LiHoF_4$. At low temperatures the susceptibility deviates from a simple Curie-type law. They found that the temperature dependence is well fitted by a power law $\chi\sim T^{-\alpha}$, with $\alpha=0.75$. The key observation made by Ghosh *et al.* (2003) is that the experimental data at low temperatures cannot be explained by simply resorting to a classical approximation. By itself this might not be enough. It is remarkable, however, that these authors are able to put in close connection the power-law divergence of the susceptibility with the entanglement

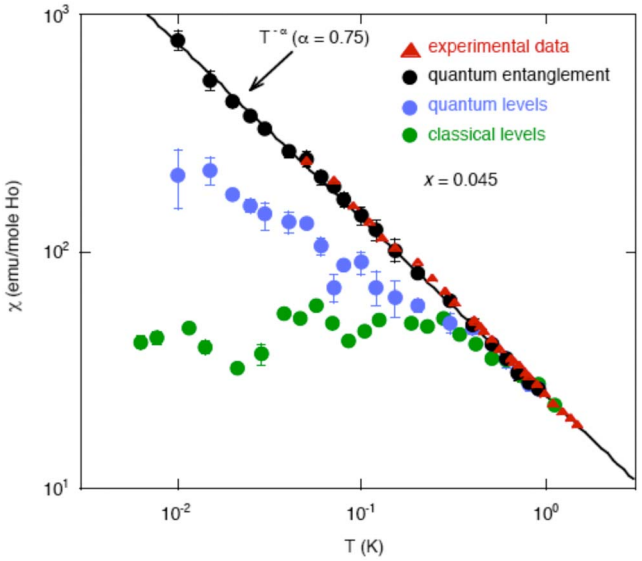

FIG. 19. (Color online) Magnetic susceptibility vs temperature T from simulations of the diluted, dipolar-coupled Ising magnet, compared to experimental data (triangles). The solid circles use quantum decimation as well as the correct quantum mechanical form of susceptibility, utilizing the entanglement of the low-lying energy doublet with the excited states. From Ghosh *et al.*, 2003.

present in the low-lying excited states; see Fig. 19.

Vértesi and Bene studied the magnetic susceptibility of NaV_3O_7 and used macroscopic entanglement witnesses as discussed in the previous section to estimate the critical temperature below which thermal entanglement is present. The experimental value of this temperature is 365 K approximately three times higher than the critical temperature corresponding to the vanishing of bipartite entanglement (Vertési and Bene, 2006). We also mention the recent experiment reporting on macroscopic magnetic measurements of the pyroborate $MgMnB_22O_5$ and warwickite $MgTiOBO_3$ [Rappoport *et al.*, 2007; see also Continentino (2006)].

Earlier experimental data witnessing entanglement in bulk properties of solids have been reanalyzed by Brukner *et al.* (2006). They discussed the experimental results of neutron scattering measurement of CN (Xu *et al.*, 2000) and showed that they provided a direct experimental demonstration of macroscopic entanglement in solids. The experimental characterization of the dynamic spin correlations for next neighboring sites enabled them to determine the concurrence and show the existence of entanglement at moderately high temperatures (as high as 5 K). In the same work they also showed that magnetic susceptibility at zero magnetic field is a macroscopic thermodynamical entanglement witness for the class of solid states systems that can be modeled by a strongly alternating spin-1/2 antiferromagnet chain (Brukner *et al.*, 2006). The measured values for magnetic susceptibility of CN (Berger *et al.*, 1963) imply the presence of entanglement in the same temperature range (below 5 K).

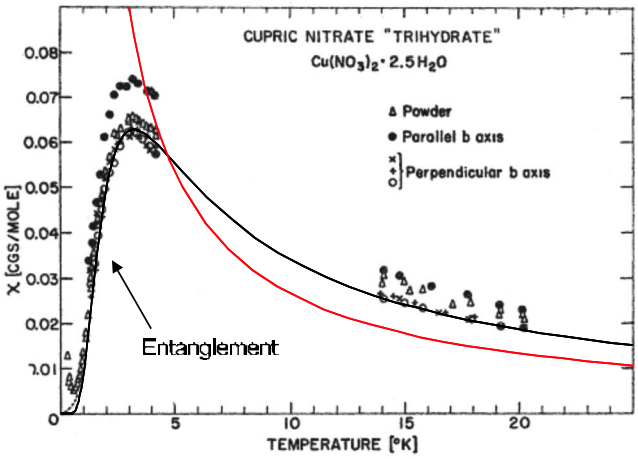

FIG. 20. (Color online) The temperature dependence of magnetic susceptibility of powder CN (triangles) and a single-crystal CN, measured at low field parallel (open squares) and perpendicular (open circles, crosses, filled circles) to the monoclinic b axis. The data and the figure are from Berger *et al.* (1963). The intersection point of this curve and the experimental one defines the temperature range (left from the intersection point) with entanglement in CN around ≈5 K. From Brukner *et al.*, 2006.

An analysis of the experimental results of a magnetic susceptibility measurement of CN (Berger *et al.*, 1963) showing that the values measured at low temperatures cannot be explained without entanglement being present was performed by Brukner *et al.* (2006). This was based on the general proof that magnetic susceptibility of any strongly alternating antiferromagnetic spin-1/2 chain is a macroscopic thermodynamical entanglement witness. As discussed by Brukner *et al.* (2006) the magnetic susceptibility for separable states is bounded by

$$\chi_{\text{sep}} \geqslant \frac{g^2 \mu_B^2 N}{T} \frac{1}{6}. \tag{67}$$

The results of their analysis are reported in Fig. 20.

VIII. MULTIPARTITE ENTANGLEMENT

Most of the results reviewed in the previous section are for pairwise entanglement. Although much has been learned from the study of those quantities, the structure of entanglement in many-body systems is much richer and it is natural to expect that multipartite entanglement is present both in the ground state and at finite temperatures. Although multipartite entanglement in many-body systems is much less studied, some important results have been already obtained.

A number of groups showed that in certain limits the state of a spin chain can resemble that of known multipartite states. For small chains Wang first noted that the ground state tends to have multipartite entanglement (Wang, 2002a). This analysis was further pursued in Ising and Heisenberg rings where the ground state has GHZ- (Stelmachovič and Bužek, 2004) and W-like (Bruß *et al.*,

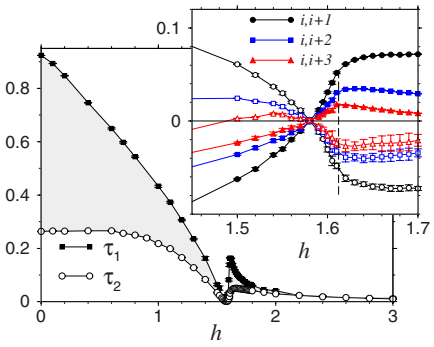

FIG. 21. (Color online) One-tangle τ_1 and the sum of squared concurrences τ_2 as a function of the applied magnetic field in the z direction for the XYZ model with exchange along y: $J_y=0.25$ (in unit of exhange along z). Inset: Contributions to the concurrence between jth neighbors; closed symbols stand for $C^{\mathrm{I}}(j)$ and open symbols for $C^{\mathrm{II}}(j)$. The dashed line marks the critical field h_c. From Roscilde *et al.*, 2004.

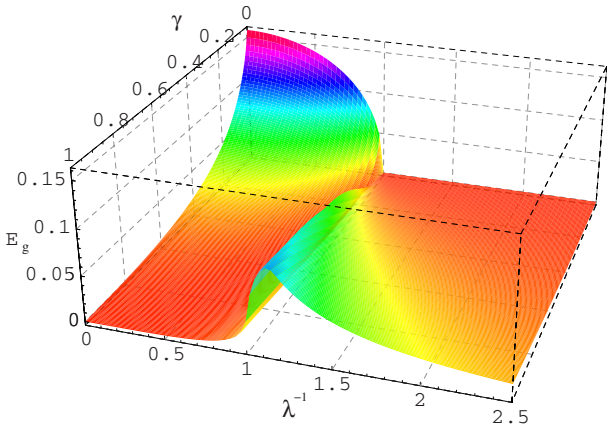

FIG. 22. (Color online) The geometric entanglement measure Eq. (12) per site versus γ and h_z for the XY model. There are three different phases: the ordered oscillatory phase for $\gamma^2+h_z^2<1$ and $\gamma\neq 0$, the ordered ferromagnetic phase between $(\gamma^2+h)z^2>1$ and $h<1$, and the paramagnetic phase for $h_z>1$. There is a sharp rise in the entanglement across the quantum phase transition line $\lambda^{-1}=h_z=1$. At $\gamma=0$ lies the XX model, which belongs to a different universality class than the anisotropic XY model. From Wei *et al.*, 2005.

2005) correlations, respectively. Multipartite mixed states can be also realized in the case of ferromagnetic rings where the ground state is fully polarized along the direction of the field and the first excited state is a W state. At finite but low temperatures the density matrix is given approximately as $\rho=p|0\cdots0\rangle\langle0\cdots0|+(1-p)|W\rangle\langle W|$, where p is the Boltzmann factor and $|0\cdots0\rangle$ is the ferromagnetic ground state. For three qubits ρ has been shown to contain tripartite entanglement by Bruß and Macchiavello (2003).

These examples show that several models naturally have multipartite entangled ground states. At the same time it also shows that it is important to quantify multipartite entanglement in many-body systems. Analysis in this direction is reviewed below.

A. Multipartite entanglement in spin systems

A first way to estimate multipartite entanglement in a spin system is provided by the entanglement ratio τ_2/τ_1 as the amount of two spin relative to global entanglement. For 1D XYZ models it was shown that a small value of such a ratio is generic in these systems with a cusp minimum at the quantum critical point (Roscilde *et al.*, 2004). This is shown numerically for the XYZ chain in a field (Fig. 21).

In order to *quantify* the multipartite entanglement, Wei *et al.* calculated the geometric measure of entanglement (Wei and Goldbart, 2003), see Sec. II.E, for the transverse XY chain (Wei *et al.*, 2005). The calculation can be done by a clever use of symmetries: translational invariance and periodic boundary conditions. In this case, the set of all possible separable states can be described by a global rotation around the y axis of the fully polarized state. The maximization is thus reduced to only one variable.

As well as in the case of bipartite entanglement, the multipartite measure of Wei *et al.* is sensitive to the existence of QPTs. As a paradigmatic example these authors analyzed the phase diagram in the anisotropy-magnetic field plane. Their results are shown in Fig. 22. As expected there is no divergence in the measure itself but in its derivative. The new aspect here is that different from the concurrence the multipartite entanglement measure in Eq. (12) can be expanded as a function of n-point correlators.

The geometric entanglement cannot discriminate between different n-particle entanglement classes. A comprehensive classification in spin systems has been given by Gühne *et al.* (2005) via the concept of k-producibility (see Sec. II.E). The systems analyzed in detail are the one-dimensional XY and Heisenberg models. Different types of n-particle quantum correlated states lead to distinct bounds for the internal energy (or the ground state energy at $T=0$). Violation of these bounds implies the presence of certain k-party producible entanglement in the system. As pointed out by Gühne *et al.* (2005), aiming at the thermodynamic limit of an infinite number of spins, the notion of k-producibility is easier to handle than the n-separability (see Sec. II.E) as its definition is independent of the number of sites in the system.

A systematic approach for deriving energy bounds for states without certain forms of multipartite entanglement has been developed by Gühne and Toth (2006). The method allows one to investigate higher dimen-

TABLE I. Energy bounds for one-, two-, three-, and four-producible states

$-4\langle H\rangle/J$	$1p$	$2p$	$3p$	$4p$
$1d$	1	3/2	1.505	1.616
$2d$	1	13/12	1.108	1.168
$3d$	1	31/30	1.044	1.067

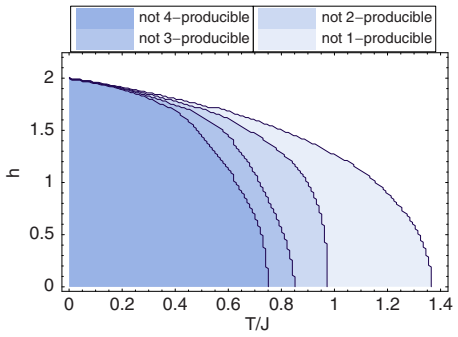

FIG. 23. (Color online) Entanglement in thermal states of the XX model in a transverse magnetic field. The different regions correspond to different types of multipartite entanglement contained in the equilibrium thermal state. From Gühne and Toth, 2006.

sional and frustrated systems. As an example we report the results for the Heisenberg model. In d dimension, the energy bounds per bond for one-, two-, three-, and four-producible states are given in Table I. It is striking how relatively close the 2- and 3-producible bounds are in all cases. All bounds given above are found to be violated in the ground state. In the previous expression the superscripts refer to the dimensionality of the model and the subscripts to the k-party entanglement for which the bound is obtained. There is a factor of $1/4$ of difference with respect to the original paper because of the different notation used in this review.

Corresponding to the energy scales fixed by the bounds there are different temperature scales at which the various n-party entanglement types disappear. As one would expect, these crossover temperatures are monotonic in k, i.e., $T_k \geqslant T_{k+1}$. The example given in Fig. 23 shows how higher order multipartite entanglement progressively disappears as the temperature increases.

B. Global entanglement

Multipartite entanglement close to quantum phase transitions was quantified by the global-entanglement E_{gl} measure of Meyer and Wallach (see Sec. II.E) by de Oliveira, Rigolin, and de Oliveira (2006); de Oliveira, Rigolin, *et al.* (2006); and Somma *et al.* (2004). Together with the Meyer-Wallach measure, de Oliveira *et al.* also introduced a slight extension of it as

$$E_{\mathrm{gl}}^{(2)} = \frac{4}{3}\frac{1}{N-1}\sum_{l=1}^{N-1}\left[1 - \frac{1}{N-1}\sum_{j=1}^{N}\mathrm{Tr}\,\rho_{j,j+l}^{2}\right], \tag{68}$$

where $\rho_{j,j+l}$ is the reduced density matrix associated to sites j and $j+l$. Similarly one can consider three-body reduced density matrices and construct the corresponding global entanglement measure. According to de Oliveira *et al.* the above hierarchy provides a comprehensive description of entanglement in many-body systems. Explicit calculations for the one-dimensional Ising model (de Oliveira, Rigolin, and de Oliveira, 2006) showed that both E_{gl} and $E_{\mathrm{gl}}^{(2)}$ are maximal at the critical point (with $E_{\mathrm{gl}} < E_{\mathrm{gl}}^{(2)}$ for any value of the coupling constant) suggesting that the quantum critical point is characterized by the presence of multipartite entanglement. As in the case of concurrence the nonanalyticity associated with the critical point is manifest in the derivative of the global entanglement measure. Extending an earlier approach developed by Wu *et al.* (2004), de Oliveira *et al.* also showed how the nonanalytic behavior of $E_{\mathrm{gl}}^{(n)}$ is related to that of the ground state energy. Note that from Eq. (68) it is possible to define an entanglement length proportional to the correlation length ξ. This differs considerably from the one defined by the localizable entanglement [see Eq. (55)]; the latter is always bounded from below by the correlation length and can even be divergent where ξ is finite.

As discussed by Facchi *et al.* (2006, 2007) the analysis of the average purity might not be sufficient and analysis of the purity distribution for different partitions could give additional information. Rather than measuring multipartite entanglement in terms of a single number, one characterizes it by using a whole function. One studies the distribution function of the purity (or other measures of entanglement) over all bipartitions of the system. If the distribution is sufficiently regular, its average and variance will constitute characteristic features of the global entanglement: the average will determine the "amount" of global entanglement in the system, while the variance will measure how such entanglement is distributed. A smaller variance will correspond to a larger insensitivity to the choice of the bipartition and, therefore, will be characteristic for different types of multipartite entanglement. Application of this technique to the one-dimensional quantum Ising model in a transverse field shows that the distribution function is well-behaved and its average and second moment are good indicators of the quantum phase transition (Costantini *et al.*, 2007). This is in agreement with previously obtained results. At the onset of the QPT both the average and the standard deviation exhibit a peak that becomes more pronounced as the number of qubits is increased.

C. Generalized entanglement

A different route for studying the multipartite entanglement is paved by the general observable based entanglement (see Sec. II.E). It was first pursued by Somma *et al.* for the LMG and the quantum XY model. In the realm of solid state systems an experimental protocol to measure many-fermion entanglement based on this concept has been proposed by Kindermann (2006). A connection which emerges the work of Somma *et al.* is the one between the generalized entanglement and quantum fluctuations of the magnetization which are responsible for the quantum phase transition (Somma *et al.*, 2004). Later, Montangero and Viola considered the dynamical behavior of generalized entanglement in disordered systems (Montangero and Viola, 2006). As remarked by Somma *et al.* (2004), it is important to choose

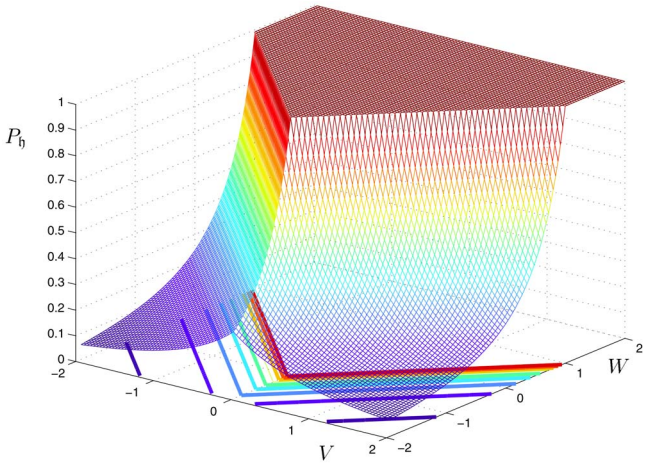

FIG. 24. (Color online) Purity relative to the observable S^z in the ground state of the LMG model. From Somma *et al.*, 2004

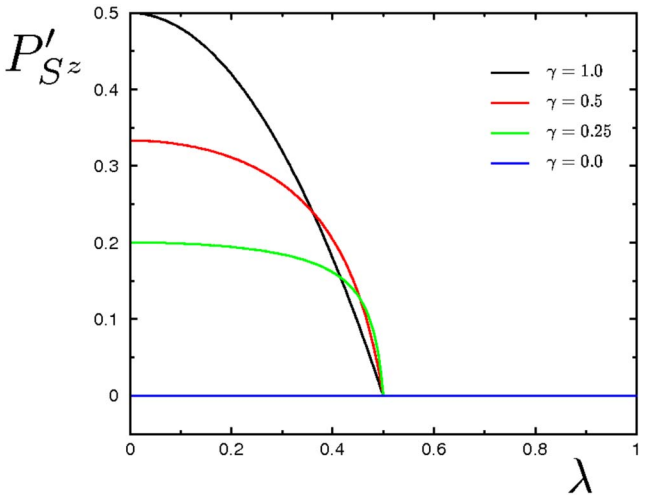

FIG. 25. (Color online) The purity $P'_{u(N)}=P_{u(N)}-(1+\gamma)^{-1}$ of an XY model in a transverse field as a function of λ for different anisotropies γ. The constant part has been subtracted in order to make the connection to the *disorder parameter*. From Somma *et al.*, 2004.

the appropriate subset of observables in order to see the critical behavior in the entanglement.

In the case of the LMG model a natural choice for the local observables was to consider the purity relative to the spin-$N/2$ representation of the angular momentum $P_S=(4/N^2)[\langle S^x\rangle^2+\langle S^y\rangle^2+\langle S^z\rangle^2]$. Somma *et al.* also considered the purity relative to the single observable S_z: $P_{S^z}=(4/N^2)[\langle S^z\rangle^2]$. With this last choice the relation between the multipartite entanglement and the order parameter becomes evident. The result is shown in Fig. 24.

Further interesting results emerge from the XY model. By choosing the following subset of operators expressed in terms of the spinless fermions of the Jordan-Wigner transform $u(N)=\text{span}\{c_i^\dagger c_{i'}+c_{i'}^\dagger c_i, i(c_i^\dagger c_{i'}-c_{i'}^\dagger c_i), \sqrt{2}(c_i^\dagger c_i-1/2)\}$ it is possible to show that the associated purity may be considered as a disorder parameter, i.e., it is nonzero in the symmetric phase only. A transparent way to express the purity in this case is to relate it to the variance of the number fermion operator $N_f=\Sigma_i c_i^\dagger c_i$. The result is plotted in Fig. 25 for different values of the anisotropy parameter γ. Two considerations are in order at this stage. First, it is clear from the previous example that important properties of entanglement appear when one moves away from the conventional picture of partitioning in real space. Second, the purity shows different from other measures as the concurrence, direct indications of critical behavior (and not only in the derivatives).

D. Renormalization group for quantum states

We continue our discussion on multipartite entanglement with recent work (Verstraete *et al.*, 2006) where the method of renormalization group was applied to quantum states and not, as traditionally done, to the Hamiltonian. The renormalization group is based on a recursive transformation which leads to an effective description of the low-energy (long distance) physics of a given system where all effects of high energy modes are included in a renormalization of certain coupling constants of the model Hamiltonian. A flow study of these recursive equations, their fixed point(s), and the behavior of the flow close to the fixed points allows one to determine the critical behavior of the system under consideration. Contrary to the conventional renormalization group approach, the idea of Verstraete *et al.* is to analyze how the quantum states change under this coarse graining and to classify their fixed points. Given a system characterized by N sites and a D-dimensional local Hilbert space. The steps of this *real space* renormalization procedure are the following.

(i) Coarse graining, in which clusters of neighboring sites are merged into one site of a new lattice.

(ii) Rescaling of the local variables associated to the new sites.

(iii) Identification of the states which are invariant under local unitary transformation (long distance behavior should not depend on the choice of the local basis).

(iv) Rescale the distances in order to have a unit lattice constant.

In this way a representative of a given class of quantum states, invariant under local transformations, is transformed at each step. The irreversibility of the transformation is crucially related to step (iii) of the transformation as one loses track of the unitary transformation performed (which needs to be local over the cluster).

Verstraete *et al.* carried out explicitly the renormalization group transformation for matrix product states (MPS), see Eq. (29). The decimation step in which two neighboring sites, say i and $i+1$, are merged together is performed by merging the corresponding matrices A^{p_i} and $A^{p_{i+1}}$ into $\tilde{A}^{p_i p_{i+1}}_{\alpha\beta}=\Sigma_{\gamma=1}^{\min(D^2_{\text{MPS}},D^2)}A^{p_i}_{\alpha\gamma}A^{p_{i+1}}_{\gamma\beta}$. With a singular value decomposition of $\tilde{A}$ it is possible to find the representative of the new state [see step (iii)] and therefore to iterate the renormalization group map. In the

case $D_{\mathrm{MPS}}=2$, Verstraete *et al.* provided a complete classification of the fixed point transformation which contains product, GHZ, and domain wall states. A similar analysis in the case $D_{\mathrm{MPS}}>2$ and/or in higher dimensional systems may turn out to be useful for the classification of multipartite entanglement in many-body systems.

E. Entanglement distribution for Gaussian states

It has been observed in Sec. II.G that the symmetry groups of admissible local transformations of Gaussian states and qubits are isomorphic. This suggests to look for further analogies with the qubit case or to search for deviations from it. A striking feature of qubit entanglement is the monogamy as far as entanglement distribution along chains of qubits is concerned (see Sec. II.B). For continuous variable entanglement such an inequality was originally analyzed analytically for fully symmetric Gaussian states and numerically for randomly chosen Gaussian states by Adesso and Illuminati (2005a). The general proof of monogamy has been obtained by Hiroshima *et al.* (2007). In this section we use the particular case of a symmetric state as a guideline to discuss the monogamy for Gaussian states. A more detailed discussion and the general proof can be found in the review by Adesso and Illuminati (2007).

A Gaussian state is called fully symmetric if and only if its covariance matrix is invariant with respect to permutations of the modes. Its covariance matrix can then be written as (Adesso *et al.*, 2004a)

$$V_{\mathrm{symm}}=\begin{pmatrix} \hat{\beta} & \hat{\varepsilon} & \cdots & \hat{\varepsilon} \\ \hat{\varepsilon} & \hat{\beta} & \cdots & \vdots \\ \vdots & \hat{\varepsilon} & \ddots & \hat{\varepsilon} \\ \hat{\varepsilon} & \hat{\varepsilon} & \cdots & \hat{\beta} \end{pmatrix}, \tag{69}$$

where $\hat{\beta}$ and $\hat{\varepsilon}$ are 2×2 matrices. Both can be diagonalized by means of local symplectic transformations in phase space, such that $\hat{\beta}=\mathrm{diag}\{b,b\}$ and $\hat{\varepsilon}=\mathrm{diag}\{\varepsilon_1,\varepsilon_2\}$. This leads to a highly degenerate symplectic spectrum where $n-1$ doubly degenerate symplectic eigenvalues are equal to $\nu=(b-\varepsilon_1)(b-\varepsilon_2)$ and the remaining eigenvalue is $\nu_n=[b+(n-1)\varepsilon_1][b+(n-1)\varepsilon_2]$. After a partial transposition, $n-2$ symplectic eigenvalues ν remain unaffected. The negativity is then determined solely by $\tilde{\nu}_-$, which is the smaller one of the remaining two affected eigenvalues $\nu_\pm$ (Adesso *et al.*, 2004a). This is due to the fact that the uncertainty leads to a lower bound $\hbar^2/4$ for the product $\nu_+\nu_-$.

For a single Gaussian mode with covariance matrix V_1, the tangle is given by $\tau_1=2(1-\det V_1)/\sqrt{\det V_1}$. The quantity which is analog to the pairwise tangle has been identified as the square of the logarithmic negativity,

$$\tau^2(\tilde{\nu}_-)=[\ln \tilde{\nu}_-]^2, \tag{70}$$

and has been termed contangle by Adesso and Illuminati (2005a). Identification of the square negativity as the continuous variable tangle is crucial for the demonstration of the monogamy inequality. Extensions of these measures to mixed states are given by the corresponding convex roofs (Uhlmann, 2000), where the average of the pure state measure of entanglement has to be minimized over all pure state decompositions of the density matrix. Restriction to decompositions purely out of Gaussian states gave rise to the notion of the Gaussian entanglement measures (Wolf *et al.*, 2004) and the Gaussian entanglement of formation. It clearly establishes an upper bound for the entanglement of formation.

Negativity and Gaussian entanglement measures have been analyzed by Adesso and Illuminati (2005b) for two-mode Gaussian states with particular focus on symmetric Gaussian states. One important result is that negativity and Gaussian measures lead to different ordering of Gaussian states with respect to their entanglement. For symmetric Gaussian states instead, both give the same ordering. This result must be handled with care, since it is not obvious what precisely the restriction to Gaussian decompositions entails for the ordering of states. Believing in the striking similarity to qubit systems, one might be tempted to conjecture that for symmetric Gaussian states a purely Gaussian optimal decomposition always existed.

There are two extremal classes of two-mode Gaussian states considered: for fixed local and global purity, those states that maximize the negativity are termed Gaussian maximally entangled mixed states (GMEMS) introduced by Adesso *et al.* (2004b), whereas those states minimizing the negativity have been termed GLEMS, which are states whose covariance matrix has one symplectic eigenvalue equal to 1 (mixed states with partial minimal uncertainty). The symmetric two-mode Gaussian states have been singled out as those states containing minimal Gaussian entanglement at fixed negativity (Adesso and Illuminati, 2005b).

The entanglement sharing inequality for Gaussian states assumes the same form as for qubits $\tau_{1,i}\geqslant\Sigma_{j\neq i}\tau_{2,ij}$, where the indices are the numbers of the various distinguishable modes in a multimode Gaussian state. This inequality has been proven for multimode Gaussian states by Hiroshima *et al.* (2007). In the particular case of symmetric states $\tau_{1,i}\equiv\tau_{1,1}$ and $\tau_{2,ij}=\tau_{2,12}$ for all indices i,j.

As for qubit systems one can define a residual contangle by the difference $\tau_n:=\tau_{1,i}-\Sigma_{j\neq i}^{n}\tau_{2,ij}$ as a quantifier of the multipartite entanglement present in a Gaussian state. In particular, the residual contangle is an entanglement monotone under Gaussian LOCC. An important difference, however, arises as compared to the residual tangle for qubits: not even for three modes is the residual tangle invariant under permutations of the modes and its minimum with respect to the common mode i has to be taken. Even for symmetric Gaussian states, where this anomaly is absent, a *promiscuous* nature of continuous variable entanglement is encountered, in sharp contrast to the *monogamy* inherent to qubit entanglement (Coffman *et al.*, 2000; Yu and Song, 2005; Osborne and

Verstraete, 2006). The term promiscuous is an interpretation of the fact that the maximal residual contangle τ_3 in a symmetric Gaussian three-mode state without pairwise contangle τ_2 is smaller than the maximum possible residual contangle without this restriction. Having in mind the entanglement sharing inequality, this implies that the local contangle τ_1 is larger when τ_2 and τ_3 coexist. It is worth noting at this point that the peculiarity of Gaussian states is completely described by two-point correlation functions. Consequently, all types of multipartite entanglement are inextricably related in that the same type of correlations are responsible for either type of quantum correlation. This is not the case for non-Gaussian states and in particular not for qubit systems or general higher-dimensional local Hilbert spaces.

IX. DYNAMICS OF ENTANGLEMENT

XX,XY

The interest in studying the properties of entanglement in a many-body system has been recently directed also to the understanding of its dynamical behavior. Entanglement dynamics has been studied from different perspectives. In a spirit similar to the study of propagation of excitations in condensed matter systems, several works analyzed the propagation of entanglement starting from a given initial state where the entanglement has been created in a given portion of the many-body system. One can imagine, for example, to initialize a spin chain such that all spins are pointing upwards except two neighboring spins which are entangled. Due to the exchange interaction the entanglement, initially localized on two neighboring sites of the chain, will spread. This propagation is ballistic in clean systems. A "sound velocity" for excitations results in a finite speed for the propagation of excitations. If some weak disorder is present one might expect diffusion. Entanglement localization and chaotic behavior can eventually also be observed. An alternative approach is to start with the ground state of a Hamiltonian H_0 and let the Hamiltonian change in time. Most of the attention up to now has been devoted to the case of sudden quench, i.e., after the preparation the Hamiltonian suddenly changes to H_1. Moreover, since we are mostly dealing with interacting systems, entanglement can be also generated or it can change its characteristics during the dynamical evolution.

Another important aspect of entanglement dynamics is the possibility to generate entangled states with given properties by taking advantage of interactions present in many-body systems. This is the natural generalization of the case where a Bell state can be obtained by letting two qubits interact for a fixed time by means of an exchange coupling of XX symmetry. In the same spirit one can think to generate three-bit entangled GHZ or W states or multipartite entangled states by tailoring the appropriate exchange couplings in spin networks.

A. Propagation of entanglement

1. Pairwise entanglement

The simplest situation which we consider first is the propagation of entanglement in the one-dimensional

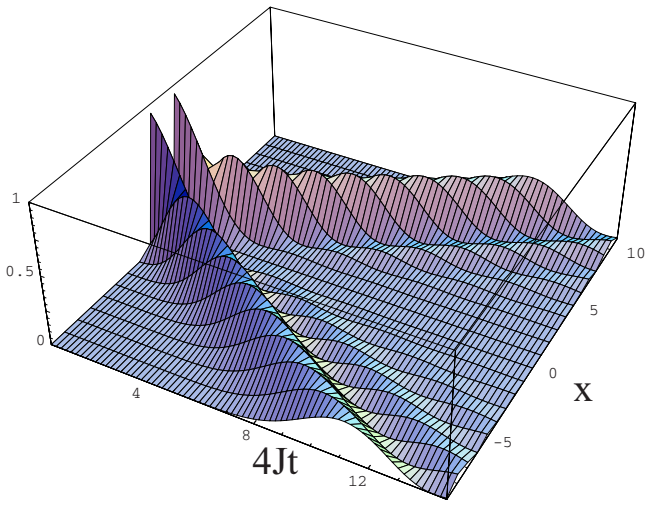

FIG. 26. (Color online) Concurrence between sites $n=-x$, $m=x$, symmetrically placed with respect to the state where the singlet was initially created. From their initial position $i=-1$ and $j=1$; From Amico *et al.*, 2004.

XX model, i.e., $\gamma=0$ and $\Delta=0$ in Eq. (23) (Amico *et al.*, 2004; Subrahmayam, 2004). Suppose the initial state of the chain is

$$|\Psi_{\pm}(t=0)\rangle \equiv \frac{1}{\sqrt{2}}(\sigma_i^x \pm \sigma_j^x)|0,\ldots,0\rangle, \tag{71}$$

namely, all spins are in a fully polarized state except two, at positions i and j, which are in one of the two Bell states $|\psi_{\pm}\rangle=2^{-1/2}(|01\rangle\pm|10\rangle)$. In this case the problem is amenable of a simple analytical solution. The reason is that, since the total magnetization is conserved, evolution is confined to the sector where only one spin is up. In this sector the dynamics is completely described by the states $|\boldsymbol{l}\rangle\equiv|0,\ldots,0,1,\ldots,0\rangle$ $(l=1,\ldots,N)$ which represents a state of the chain where the lth spin is prepared in $|1\rangle$ while all the other $N-1$ ones are in $|0\rangle$. At later times the state of the chain is to be

$$|\Psi_{\pm}(t)\rangle = \sum_l w_{\pm,l}^{(i,j)}(t)|\boldsymbol{l}\rangle. \tag{72}$$

In the thermodynamic limit $N\to\infty$ the coefficients can be expressed in terms of Bessel functions $J_n(x)$ as $w_{\pm,l}^{(i,j)}(t)=\frac{1}{\sqrt{2}}\{J_{i-l}(4Jt)\pm(-i)^{j-i}J_{j-l}(4Jt)\}$. Equation (72) together with the coefficients defined previously allows one to study various kinds of entanglement measures for this particular case.

As a first indication of the entanglement propagation we analyze the dynamical evolution of the concurrence between two sites, located at positions n and m (initially the entangled state is between sites i and j). The concurrence reads

$$C_{n,m}^{i,j}(\pm,t) = 2|w_{\pm,n}^{(i,j)}(t)w_{\pm,m}^{(i,j)\star}(t)|. \tag{73}$$

The function $C_{n,m}^{i,j}(t)$ plotted in Fig. 26 shows sites which are symmetric with respect to the initial position of the Bell state $|\psi_{\pm}\rangle$. The time evolution dictated by the XX

Hamiltonian amounts to a propagation of the single flipped spin through the chain. The speed of propagation is set up by the interaction strengths. The information exchange or entanglement propagation over a distance of d lattice spacings takes time $t \sim \hbar d/J$. The external field h does not enter into Eq. (73), since all components of the state are in the same sector where one spin has been flipped and therefore it contributes only to an overall dynamical phase to the state evolution.

Having in mind the use of spin chains as quantum channels, the preparation scheme described above does not lead to faithful entanglement transfer. The most natural way to perform this task is to prepare the entangled state and then let only one of the qubits propagate along the spin chain [thereby following the protocol originally proposed by (Bose 2003) or its modified version (Christandl *et al.*, 2004) to achieve perfect transfer with a modulated chain]. A detailed analysis in this direction was recently performed by Hartmann *et al.* (2006). They considered initial maximally entangled states and used the chain to transfer the state of one of the two qubits, found a relation between a measurement of the entanglement fidelity at the fidelity of state transfer, and concluded that there are possibilities to have perfect entanglement transfer.

If the chain is initially prepared in $|\Psi_{\pm}(t=0)\rangle$, given in Eq. (71), the entanglement propagates maintaining its original characteristics. This is not the case if, for example, the initial states of the two entangled sites is of the type $|\Phi_{\pm}\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$. These states are superpositions of components belonging to different magnetization sectors. The entanglement propagates with the same velocity as before, however, under certain conditions, it turns out that the propagating quantum correlations change their character. After some time a singlet-like entangled state propagates even if the initial state was not a singlet. A different set of initial states has also been considered. In Hamieh and Katsnelson (2005) the chain was initialized in a separable state by means of a set of projective measurement (in particular they considered all spins aligned in the z direction except one prepared in an eigenstate of S_x). The evolution can be described using the same approach as outlined above. The new ingredient now is the creation of entanglement during the dynamics. Hamieh and Katsnelson showed that oscillations of the entanglement wave have the same periodicity, but out of phase, with the oscillation of the (nonequilibrium) average magnetization. The distribution of entanglement in the chain has been analyzed after launching a single excitation from the central site of a XX chain by Fitzsimons and Twamley (2005) It was found that the second-order moment of the spatial extent of the concurrence grows much faster (with a rate increasing as $t^{5/2}$) if some disorder is present in the central portion of the chain (in the ordered case the increase goes as t^2).

!?. Lieb Robinson ?

Additional features emerge in the quantum XY with $\gamma \neq 0$. In this case the magnetization is no longer a constant of motion (two spins can be flipped simultaneously). Calculations were done analytically (Amico *et al.*, 2004) resorting to the exact set of out-of-equilibrium correlation functions (Amico and Osterloh, 2004). The most notable difference in the two-site entanglement studied is an entanglement production from the vacuum. This occurs uniformly along the chain and it superimposes to the associated entanglement wave discussed before. The velocity of propagation of the entanglement is almost independent of the anisotropy parameter γ. What is strongly dependent on γ is the damping coefficient of the entanglement wave. As the anisotropy approaches $\gamma=1$, the Ising point, the wave is strongly damped and vanishes approximately after a few $\sim J^{-1}$. Furthermore, in the anisotropic case it is possible to observe nontrivial dynamics when the external magnetic field is time dependent [Huang and Kais, 2005; Sen(De) *et al.*, 2005]. In Sen(De) *et al.* (2005) the system is initially prepared in the ground state; the evolution is analyzed after the magnetic field is (suddenly) switched off. Sen(De) *et al.* analyzed the two-site entanglement by means of the logarithmic negativity as shown in Fig. 27 and demonstrated the existence of a dynamical phase transition, not observable in the magnetization. As it can be seen in Fig. 27 at a fixed time t the entanglement vanishes (and then grows again) for certain values of the initially applied magnetic field. For $t \leqslant 1.8$ the critical field is almost independent on the time t and is $h \sim 0.8$. A remarkably different situation occurs for $t \leqslant 1.8$, here a dynamical phase transition occurs where the entanglement vanishes for any value of $h \geqslant h_c$. For values of the initial field in the region near the phase transitions, entanglement behaves nonmonotonically with respect to temperature of the initial equilibrium state.

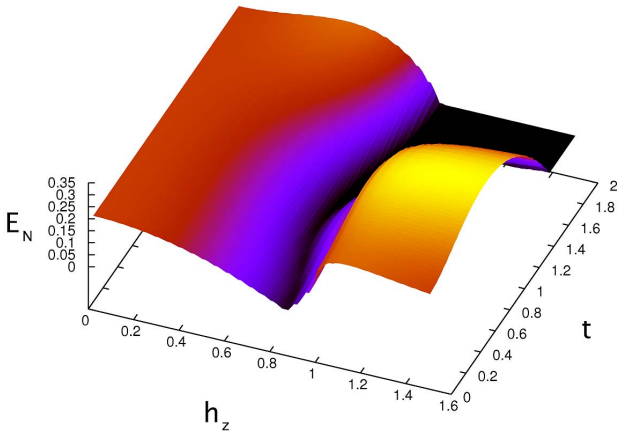

FIG. 27. (Color online) The nearest-neighbor logarithmic negativity E_N as a function of the initial transverse field h_z and time t for the anisotropy $\gamma=0.5$. At short times $t \leqslant 1.8$ the dynamical phase transition is a point of reentrance for the entanglement. At later times the state remains separable for values of the field above the critical value. For $t=0$, entanglement vanishes, as it should, as h_z grows. The transverse magnetization of the evolved state does not show a similar critical behavior as a function of h_z. From Sen(De) *et al.*, 2005.

The two-site entanglement is nonergodic [Sen(De) *et al.*, 2004]. The evolution of two-site entanglement of a XY chain was studied after a sudden change of the external magnetic field. The evolved state does not approach its equilibrium value. That is, entanglement by itself does not relax to its equilibrium value in the absence of further contact with reservoirs. Therefore entanglement in such systems cannot be described by equilibrium statistical mechanics.

The entanglement dynamics has been studied, to a large extent analytically, in the LMG model (Vidal, Palacios, and Aslangul, 2004; Latorre, Lütken, *et al.*, 2005). They considered the dynamical evolution of entanglement by analyzing both the one-tangle $\tau_1(t)$ and the concurrence $C(t)$ for cases in which the initial state is fully polarized either in the z or in the x direction. Because of the model symmetries and since the initial state belongs to the sector with maximal spin, $S=N/2$, both quantities do not depend on the spins which are selected. This means that they can be expressed in terms of the average value of the total spin $\langle S^\alpha\rangle$ and its correlation functions $\langle S^\alpha S^\beta\rangle$ $(\alpha=x,y,z)$. An interesting feature of this model is that its level spacing is finite even in the thermodynamic limit. Contrary to expectations, however, the Poincaré time is *dependent* on the number of spins N and this has important consequences on the entanglement dynamics.

When the spins are prepared in the state $|0\cdots0\rangle$ analytical results can be obtained in the limit of zero and very large magnetic field. In the limit of zero external field the tangle and the concurrence can be expressed as $\tau_1(t)=1-c(t)^{2(N-1)}$ and $C(t)=(1/4)\{c(t)^{(N-2)}-1+[c(t)^{(N-2)}-1]^2+[4c(t)^{(N-2)}s(t)^{(N-2)}]^{2^{1/2}}\}$, with $c(t)=\cos(4Jt/N)$, $s(t)=\sin(4Jt/N)$, and show perfect anticorrelation in time. In the opposite limit of a strong applied field it is possible to resort to a semiclassical approximation of the Heisenberg equation of motion (Law *et al.*, 2001), $\dot{S}_x=2hS_y$, $\dot{S}_y=-2hS_x+(2J/N)[S_z,S_x]_+$, $\dot{S}_y=-(2J/N)[S_z,S_y]_+$ by noting that, for $N\gg1$, the z component of the magnetization has negligible fluctuations $[S_z(t)\sim S=N/2]$. Therefore the set of equations governing the dynamics of the system becomes linear and can be easily solved. The concurrence as a function of time for positive values of the external field evolves as $C(t)=1-c_h(t)^2-(J^2/4h_z^2)s_h(t)^2$, with $c_h(t)^2=\cos^2[\sqrt{h(h-J)}t]$ and $s_h(t)^2=\sin^2[\sqrt{h(h-J)}t]$. Dynamics of two-site entanglement was also discussed in the context of spin-bosonlike models. Ciancio and Zanardi (2006) analyzed the negativity for a two-modes Jaynes-Cummings with particular emphasis on the entanglement between two bosonic modes mediated by the qubit. The relaxation dynamics of the entanglement, quantified through the concurrence, in the spin-boson model was discussed by Lim *et al.* (2006).

We finally mention a study where it was noted that the entanglement encoded in states caused by the splitting of the degeneracy determined by the transverse field in the quantum XY model is not preserved by an adiabatic perturbation. Separable states can become entangled after the field is varied adiabatically, and vice versa (Cao *et al.*, 2006).

2. Dynamics of the block entropy

The dynamical behavior of the block entropy was first considered by Calabrese and Cardy (2005) in the general framework of conformal field theory and via an exact solution of the quantum Ising model. Later Dür *et al.* considered Ising models with long range interaction and De Chiara *et al.* (2005) performed numerical simulations of the XXZ chain.

Calabrese and Cardy showed that a quench of the system from a noncritical to a critical point leads the block entropy first to increase linearly in time and then to saturate. For periodic boundary conditions and given a block of dimensions ℓ, the time at which the entropy saturates is given by $t^*=\ell/(2v)$, where v is the spin wave velocity:

$$S_\ell(t)\sim\begin{cases}t, & t\leqslant t^*\\ \ell, & t\geqslant t^*.\end{cases}\tag{74}$$

Thus there is an arbitrary large entanglement entropy in the asymptotic state, contrary to the ground-state case where the entropy diverges only at critical point. An example of the block entropy time dependence for the Ising and XXZ models is shown in Fig. 28. Calabrese and Cardy proposed a simple interpretation for this behavior in terms of quasiparticles excitations emitted from the initial state at $t=0$ and freely propagating with velocity v. The argument goes as follows. The initial state has a high energy relative to the ground state of the Hamiltonian which governs the subsequent time evolution, and therefore acts as a source of quasiparticle excitations. Particles emitted from different points are incoherent, but pairs of particles moving to the left or right from a given point are highly entangled (see Fig. 29). The idea is that a point x_A in the region A will be entangled with a point x_B in the region B if a pair of entangled particles emitted at an earlier time arrive simultaneously at x_A and x_B. In this picture the block entropy is proportional to the length of the interval where this occurs. Saturation is reached when the most distant quasiparticles (which started in the middle of the block) have already got entangled. In the presence of disorder the dynamical behavior is strikingly different. This picture applies in the more general context of dynamical correlation functions after a quench as discussed by Calabrese and Cardy (2006).

Possible evidence of localization in the block entropy has been discussed by De Chiara *et al.* (2005). The case of Ising models in zero magnetic field with long range interaction of the dynamics was analyzed in the case of factorized initial states (Dür *et al.*, 2005). The general picture is also confirmed in this case although in the short time limit additional oscillations appear probably due to the various different phases related to the different couplings.

We finally mention that the entropy in the case of a finite time quench has been considered by Cherng and

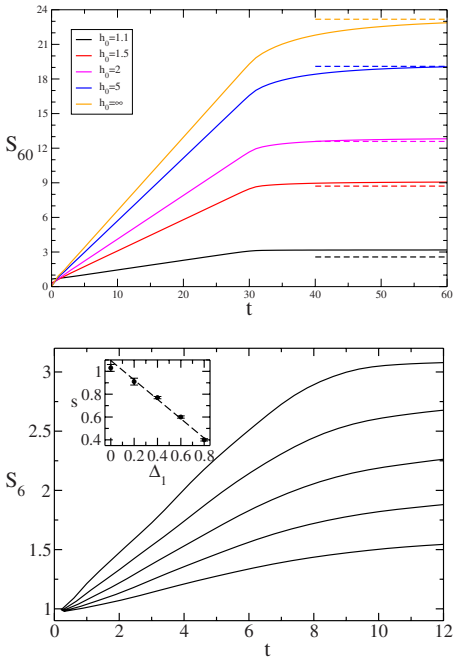

FIG. 28. (Color online) Evolution of the entropy for different types of quenches in the case of the Ising (top panel) [from Calabrese and Cardy (2005)] and Heisenberg (bottom panel) [from De Chiara *et al.* (2005)] models. The block are of 60 and 6 sites in the Ising and Heisenberg cases, respectively. In the Ising case the quench are obtained by changing the external magnetic field from h_0 to $h_1=1$. In the Heisenberg model the anisotropy parameters are used instead, $\Delta_0=1.5$ while $\Delta_1=0.0$, 0.2, 0.4, 0.6, and 0.8. The time is expressed in units of the spin wave velocity. The inset of the top panel shows the behavior of the initial slope as a function of the final value of the anisotropy.

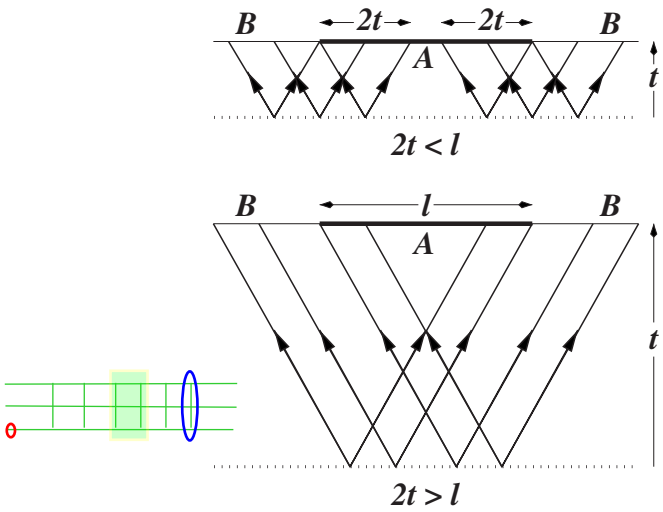

FIG. 29. Schematic representation of the of block entropy dynamics. Entangled particles are emitted from region A, and they contribute to the block entropy as long as one of the two particles ends in region B. From Calabrese and Cardy, 2005.

Levitov (2006) and by Cincio *et al.* (2007). Recently rigorous bounds for the time evolution of the block entropy were obtained by Bravyi *et al.* (2006) and by Eisert and Osborne (2006).

3. Chaos and dynamics of entanglement

The evolution of entanglement is sensitive to the different properties of the spectrum in the transition to chaos as in the case of a quantum computer with imperfections. A small inaccuracy in the coupling constants induces errors in gating or a unwanted time evolution if the Hamiltonian cannot be switched exactly to zero. If the imperfection strength increases, new phenomena occur and above a certain threshold the core of the computer can even "melt" due to the setting in of chaotic behavior (Georgeot and Shepelyansky 2000; Benenti *et al.*, 2001). In addition to understanding the behavior of the fidelity as an indicator to measure the stability of the quantum memory [see Gorin *et al.* (2006) for a review], a more complete characterization has included the behavior of entanglement on approaching the transition to chaos either by considering dynamics of a disordered [Montangero and Viola, 2006; Montangero *et al.*, 2006; Sen (De) *et al.*, 2006] or time-dependent Ising model [Lakshminarayan and Subrahmayam (2005); see Prosen (2007) for a recent review on the dynamical complexity analyzed on the kicked Ising model] and by studying the dynamics of a quantum map (Miller and Sarkar, 1999; Bandyopadhyay and Lakshminarayan, 2002; Bettelli and Shepelyansky, 2003; Ghose and Sanders, 2004; Rossini *et al.*, 2004). In particular the disordered Ising model has been proposed (Georgeot and Shepelyansky, 2000) to describe the hardware of a quantum computer, in which system imperfections generate unwanted interqubit couplings and energy fluctuations. Three different regimes appear depending on the variance of the fluctuations of the exchange couplings. At the critical value δJ_c the system crosses from the perturbative to the chaotic regime, while at the second point δJ_E the system goes into an ergodic state characterized by a Gaussian local density of states. All these dynamical regimes have been detected in the dynamics of entanglement by Montangero *et al.* (2003) and Montangero and Viola (2006).

We finally mention the interesting connections (Boness *et al.*, 2006) found for the properties of the entanglement in a Heisenberg chain with pulsed magnetic field with the localization behavior of the quantum kicked rotator.

B. Generation of entanglement

The Hamiltonians discussed so far can be also used to generate particular entangled states. The simplest case to consider is the XX model in the sector in which only one spin is flipped. The fact that the Hilbert space is spanned by the basis set $|\ell\rangle$ hints to the fact that W-states can be generated. For short chains, $N=3,4$, generalized W-states of the type $|W\rangle=\frac{1}{\sqrt{4}}[|1000\rangle-i|0100\rangle-|0010\rangle-i|0001\rangle]$ appear at discrete times as the initial state of the chain is fully polarized (Wang, 2001). This simple scheme cannot be extended easily to an arbitrary

number of qubits. The reason is related to the fact that for generic terms the various frequencies appearing in the dynamical evolution of the state are not commensurate.

An interesting example of entangled state generation is that of cluster states (Briegel and Raussendorf, 2001) which has been shown to be essential for one-way quantum computation (Raussendorf and Briegel, 2001; Raussendorf *et al.*, 2003) [see Hein *et al.* (2006) for a review]. Remarkably, they can be generated by certain spin Hamiltonians. In fact, the dynamical evolution of an Ising model in zero magnetic field, $H=\Sigma_{i,j}J_{ij}\sigma_i^z\sigma_j^z$, is equivalent to a series of conditional phase shifts. In the case of a regular lattice this Hamiltonian generates cluster states which are readily generalized to graph states for an Ising model defined on a simple graph. It is easy to realize that the evolution operator at time $Jt=\pi/4$ can be written as $U(\pi/4)=\Pi_{i,j}\frac{1}{2}(1+\sigma_i^z\sigma_j^z)$. If the initial state is a product state with all spins pointing along the x direction, then at time $Jt=\pi/4$ the state is a graph state characterized by a maximal bipartite entanglement and maximal entanglement persistence. An example of a cluster state generated by an Ising chain with nearest-neighbor couplings, for $N=4$, is given by $|\Psi\rangle_{\mathrm{cl}}=|0000\rangle+|1100\rangle+|0011\rangle-|1111\rangle$ (for $N=2,3$ cluster states coincide with Bell and GHZ states, respectively). The Ising model is not the only case when graph states can be created. Borhani and Loss (2005) showed how to generate them using the Heisenberg interaction while Clark *et al.* (2005) considered the XX-model Hamiltonian.

An appropriate tailoring of the initial state or the spin graph can lead to production of properly tailored entangled states (Koniorczyk *et al.*, 2005). In an XX model in a star network it is possible to control the concurrence between two spins by varying the initial state of the central spin only. Such a spin mediates the interaction between the outer ones as discussed by Hutton and Bose (2004). The pairwise entanglement can be maximized by choosing all outer spins down and the central spin up. These dynamically generated states saturate the Coffman-Kundu-Wootters (CKW) inequality, Eq. (8) (Coffman *et al.*, 2000) and hence have the maximal possible two-site entanglement. Koniorczyk *et al.* analyzed the concurrence of assistance, i.e., the maximum amount of entanglement which could be concentrated on two qubits by means of assisted measurements on the rest of the system. Depending on the initial system, the behavior of the concurrence and concurrence of assistance can be quite different.

C. Extraction of entanglement

The entanglement naturally contained in a many-body state can in principle be extracted and therefore used for quantum information processing. This means that such entanglement can be transferred to a pair of particles and subsequently used, in principle, for quantum computation or to test the violation of Bell's inequalities. De Chiara *et al.* proposed a scheme of entanglement swapping by scattering a pair of particles with a spin chain or an optical lattice (De Chiara *et al.*, 2006). To this end one sends simultaneously a pair of probe particles toward the entangled spin chain in such a way that each probe interacts with a different spin. The entanglement between probes has been extracted from the spin chain and cannot exist without entanglement in the chain. This is a genuine nonlocal process between two probes as in the case of entanglement swapping. In practice the scattering interaction between probes and spins in the chain must be capable of (at least partially) swapping their state. This is the case of very common interactions like the Heisenberg or the XY ones. The most natural way to extract entanglement from entangled electron spins in solids would be to scatter pairs of neutrons off the solid. Another possibility would be to use Hamiltonians of entangled spin chains or ladders that can be realized using trapped cold atoms (Duan *et al.*, 2003; Garcia-Ripoll *et al.*, 2003; Garcia-Ripoll and Cirac, 2004) and as probe particles one can use marker qubits (Calarco *et al.*, 2004).

D. Time evolution of the entanglement in Gaussian states

The dynamics of Gaussian states was first discussed by Plenio *et al.* (2004), where essentially two aspects have been highlighted: The creation of entanglement from an initially disentangled state and the propagation of an entangled state on top of a disentangled background, both induced by Hamiltonian dynamics.

The initially entangled two-mode state is characterized by the squeezing parameter r, which appears in the co-variance matrix as $V_{\xi_\alpha,\xi_\alpha}=\cosh r$ for all phase space variables of the zeroth and first oscillator modes, whereas $V_{q_0,q_1}=V_{p_0,p_1}=\sinh r$; in absence of further off-diagonal elements, all other diagonal elements are equal to 1. Two different types of nearest-neighbor couplings of the oscillator have been considered: ideal springs obeying Hooke's law and the rotating wave approximation (RWA). In this approximation both the kinetic and potential energy terms assume the same form (Plenio *et al.*, 2004). The RWA conserves the number of bosonic excitations in the system, whereas the ideal spring coupling does not. The initial entanglement is encoded in a zeroth oscillator, itself decoupled from the chain of oscillators, and one oscillator within the chain as described above. In this case, the Hamiltonian for the harmonic lattice [with an appropriate choice of the matrix $\mathbb{U}$, see Eq. (34)] reads

$$H=\frac{1}{2}\left(q_0^2+p_0^2+\sum_{k=1}^{N}[q_k^2+p_k^2+K_1(q_{k+1}-q_k)^2]\right). \quad (75)$$

After discarding the terms $\hat{a}\hat{a}$ and $\hat{a}^\dagger\hat{a}^\dagger$, in RWA, Eq. (75) can be written (up to a constant) as (Plenio *et al.*, 2004)

$$H_{\mathrm{RWA}} = \hat{a}_0^\dagger \hat{a}_0 + (1+K_1)\sum_{k=1}^{N} \hat{a}_k^\dagger \hat{a}_k - K_1(\hat{a}_{k+1}^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_{k+1}). \tag{76}$$

For both Eqs. (75) and (76) the time evolution of the position and momentum operator can be evaluated analytically.

The initially entangled state of two oscillators—a decoupled and one harmonic oscillator within a periodic chain—is released at time $t=0$ into the background of initially disentangled oscillators, all prepared in their ground state. In regard to the pairwise entanglement we observe that the nth and zeroth oscillators become entangled after a finite time $t_c(n)$, which is given by the velocity of sound of the underlying model Hamiltonian. After this entanglement wave arrives, the nth oscillators' entanglement exhibits damped oscillations with the characteristic time scale of the model. For the model (75), the velocity of sound has been determined as $v = K_1/\sqrt{1+2K_1}$. Within the RWA this velocity appears enhanced: $v_{\mathrm{RWA}}=K_1$. The height E_{Nf} of the first local maximum in the logarithmic negativity has been used to define the transmission efficiency T_{eff} for entanglement in the chain as $T_{\mathrm{eff}}=E_{Nf}/E_{Ni}$, where E_{Ni} is the logarithmic negativity initially prepared between the oscillators zero and one.

In both models E_{Nf} is observed to saturate when cranking up r and hence the initial entanglement. As expected, the saturation value is a decreasing function of the distance n. The main difference is that for generic coupling the saturation value depends on the coupling strength K_1 and the smaller it is, the stronger the coupling is. Within the RWA it is independent of the coupling strength. Given the initial logarithmic negativity $E_{Ni}=-2r/\ln 2$, the transmission efficiency behaves differently. For the generic model it is zero for zero squeezing, has a maximum at a finite squeezing $r_{\max}$, and vanishes for $r\to\infty$. In sharp contrast, in RWA T_{eff} is a monotonically decreasing function of r. Interestingly enough, for the model in RWA, the oscillator frequencies and coupling strengths can be optimized to have perfect transmission from one end of an open chain of M sites to the other end. This is achieved by virtue of $\mathbb{U}_{j,j+1} = \mathbb{U}_{j+1,j} = K_1\sqrt{n(M-n)}$, $\mathbb{U}_{j,j}=1$, and $\omega_n = 1 - K_1\sqrt{n\bar{n}} - K_1\sqrt{(n-1)(\bar{n}+1)}$, where $\bar{n}=M-n$. The same turns out to be impossible for the generic model and $M>2$. The only possibility is to choose the couplings and frequencies as in the RWA case in the limit of $K_1\to 0$ (where the RWA is exact). In this way, the transmission efficiency can be pushed arbitrarily close to perfect transmission but with a transmission speed tending to zero. The transmission of quantum information has been found to be robust against noise in K_1 for both models (Plenio *et al.*, 2004).

Another effect occurring in the generic model is entanglement creation from a disentangled state, which is not an eigenstate. This can be realized by an abrupt change of the coupling strength. As in the spin case (see Sec. IX.A) no entanglement creation can be generated within the RWA. In an open chain the oscillators at the end points become entangled after half the time a signal needs to travel through the entire chain. This indicates that this initial pairwise entanglement is mediated from the center of the chain. Actually this is the first signal that can possibly arrive when essentially pairwise entanglement is created or contributes to the eventual pairwise entanglement of the boundary oscillators. Raising a finite temperature for a thermal state, the arrival time is slightly enhanced and the signal is subject to damping. However, this effect turned out to be much more sensitive to noise in the coupling strength than the propagation of quantum information (Plenio *et al.*, 2004).

X. CONCLUSIONS AND OUTLOOK

During the last years it became evident that quantum information offers powerful instruments to grasp the properties of complex many-body systems. This is the reason why this area of research is undergoing an impressive expansion. In this review we touched only one particular aspect of this activity, namely the properties of entanglement in many-body systems.

As mentioned in the Introduction, there are several important aspects which, for space limitations, were ignored. In this respect we should certainly mention the increasing interest towards the optimization of numerical simulations of quantum systems. There were already a number of breakthroughs (see also the Introduction) that, for example, lead to the development of the time-dependent DMRG. The design of variational methods to study the ground state and finite temperature properties of many-body Hamiltonians has been exploited in numerous work. We mentioned in the Introduction the large body of activity on quantum state transfer in spin chains. Here the knowledge of low-lying excitations of the chain (spin waves) has helped in finding new quantum protocols. More is expected to come in the future.

It is tempting, although difficult, to try to envisage the possible evolution of this line of research. The study of the topological entanglement entropy is an important aspect that will be investigated in the future. Adiabatic quantum computation and one-way quantum computation will benefit from the study of entanglement in complex systems. The study of the topological entanglement may also have remarkable spin off to understand many puzzling phenomena in condensed matter physics among which high-T_c superconductors and heavy fermions are of paramount importance (Coleman and Schofield, 2005). Furthermore, analysis of new measures for multipartite entanglement may provide the additional insight necessary for understanding the role of entanglement in such complex phenomena, which might also reveal valuable information, e.g., on how to simulate these systems efficiently on a computer.

Many interesting results have been already obtained, but the overlap between quantum information and many-body physics is not yet fully unveiled. There is a

number of open questions which provide a fertile ground for a field of lively exciting research.

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