Plasma Kinetic Theory
Lecture script

September 28, 2010

This document is fully based on the plasma physics-theory book by Dwight R. Nicholson [1]. It is intended as a brief script for the lecture KINETIC THEORY IN PLASMA PHYSICS held by Winfried Kernbichler.

Contents

1 FROM THE DENSITY FUNCTIONAL TO THE DISTRIBUTION FUNCTION

Plasma kinetic theory is the theory of plasma taking into account the motions of all of the particles. The *Klimontovich* and *Liouville equation* provide exact descriptions of the plasma, however the equations are far from being solvable for practical purposes. Usually one is rather interested in certain average or approximate characteristics. The usefulness of the Klimontovich and Liouville equation is as a starting point for the derivation of approximate equations that describe the average properties of a plasma.

Suppose we have a gas consisting of only one particle. This particle has an orbit $\boldsymbol{X}_1(t)$ in 3-D configuration space $\mathbb{X}$ with coordinates $\boldsymbol{x}$. The orbit $\boldsymbol{X}_1(t)$ is the set of positions $\boldsymbol{x}$ occupied by the particle at successive times t. Likewise the particle has an orbit $\boldsymbol{V}_1(t)$ in 3-D velocity space $\mathbb{V}$ with coordinates $\boldsymbol{v}$. We combine these spaces into 6-D phase space $\mathbb{P}$ with coordinates $(\boldsymbol{x}, \boldsymbol{v})$. The *density functional of one particle* in $\mathbb{P}$ is

$$N(\boldsymbol{x}, \boldsymbol{v}, t) = \delta(\boldsymbol{x} - \boldsymbol{X}(t))\delta(\boldsymbol{v} - \boldsymbol{V}(t)). \tag{1}$$

At any time t, the density of particles integrated over all phase space must yield the total number of particles in the system.

Consider a system that contains two species of particles, electrons and ions, each with N_0 particles. The density N_s of species s is

$$N_\mathrm{s}(\boldsymbol{x}, \boldsymbol{v}, t) = \sum_{i=1}^{N_0} \delta(\boldsymbol{x} - \boldsymbol{X}_i(t))\delta(\boldsymbol{v} - \boldsymbol{V}_i(t)) \tag{2}$$

and the total density N is

$$N(\boldsymbol{x}, \boldsymbol{v}, t) = \sum_{\mathrm{e,i}} N_\mathrm{s}(\boldsymbol{x}, \boldsymbol{v}, t). \tag{3}$$

The exact positions and velocities of the particles are determined by their initial conditions because the position $\boldsymbol{X}_i(t)$ of particle i satisfies the equation

$$\dot{\boldsymbol{X}}_i(t) = \boldsymbol{V}_i(t). \tag{4}$$

Likewise, the velocity $\boldsymbol{V}_i(t)$ of particle i satisfies the Lorentz force equation

$$m_\mathrm{s}\dot{\boldsymbol{V}}_i(t) = q_\mathrm{s}\boldsymbol{E}^\mathrm{m}[\boldsymbol{X}_i(t), t] + \frac{q_\mathrm{s}}{c}\boldsymbol{V}_i(t) \times \boldsymbol{B}^\mathrm{m}[\boldsymbol{X}_i(t), t]. \tag{5}$$

The microscopic electric and magnetic fields are the fields produced selfconsistently by the point particles themselves, together with externally applied fields. The microscopic fields satisfy Maxwell's equations

$$\nabla \cdot \boldsymbol{E}^\mathrm{m}(\boldsymbol{x}, t) = 4\pi \rho^\mathrm{m}(\boldsymbol{x}, t) \tag{6}$$

$$\nabla \cdot \boldsymbol{B}^\mathrm{m}(\boldsymbol{x}, t) = 0 \tag{7}$$

$$\nabla \times \boldsymbol{E}^\mathrm{m}(\boldsymbol{x}, t) = -\frac{1}{c}\frac{\partial \boldsymbol{B}^\mathrm{m}(\boldsymbol{x}, t)}{\partial t} \tag{8}$$

$$\nabla \times \boldsymbol{B}^\mathrm{m}(\boldsymbol{x}, t) = \frac{4\pi}{c}\boldsymbol{J}^\mathrm{m}(\boldsymbol{x}, t) + \frac{1}{c}\frac{\partial \boldsymbol{E}^\mathrm{m}(\boldsymbol{x}, t)}{\partial t} \tag{9}$$

The microscopic charge density and microscopic current density is

$$\rho^\mathrm{m}(\boldsymbol{x}, t) = \sum_{\mathrm{e,i}} q_\mathrm{s} \int \mathrm{d}\boldsymbol{v}\, N_\mathrm{s}(\boldsymbol{x}, \boldsymbol{v}, t) \tag{10}$$

and

$$\boldsymbol{J}^{\mathrm{m}}(\boldsymbol{x}, t) = \sum_{\mathrm{e,i}} q_{\mathrm{s}} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} N_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{v}, t), \tag{11}$$

respectively. While Maxwell's equations determine the fields in terms of the exact particle orbits, the Lorentz equation determines the exact particle orbits in terms of the exact fields. The entire set of equations is closed, so that if the positions and velocities of all particles, and the fields, are known exactly at one time, then they are known exactly at all later times.

2 KLIMONTOVICH EQUATION

An exact equation for the *evolution of a plasma* is obtained by taking the time derivative of the density functional N_{s}. Using the relations $a\delta(a-b) = b\delta(a-b)$ and $\partial/\partial a f(a - b) = -\partial/\partial b f(a - b)$ we finally arrive at the **Klimontovich equation**,

$$\frac{\partial N_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{v}, t)}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} N_{\mathrm{s}} + \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} \left(\boldsymbol{E}^{\mathrm{m}} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}^{\mathrm{m}} \right) \cdot \nabla_{\boldsymbol{v}} N_{\mathrm{s}} = 0. \tag{12}$$

Together with Maxwell's equations, it constitutes an exact description of a plasma. The Klimontovich equation contains every one of the exact single particle orbits. What we really want instead is information about certain *average properties of the plasma*. The Klimontovich equation expresses the *incompressibility* of $N_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{v}, t)$ as it moves about in $\mathbb{P}$.

Consider the orbit of a hypothetical particle in $\mathbb{P}$. Imagine taking a time derivative of any quantity along this orbit, i.e. the convective derivative that not only considers the explicit time variation of the quantity but also the time variation produced by changing the position in phase space $\mathbb{P}$,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \left.\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|_{\mathrm{orbit}} \cdot \nabla_{\boldsymbol{x}} + \left.\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|_{\mathrm{orbit}} \cdot \nabla_{\boldsymbol{v}}. \tag{13}$$

With this notation the Klimontovich equation simply says

$$\frac{\mathrm{D}N_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{v}, t)}{\mathrm{D}t} = 0, \tag{14}$$

which means the density of particles of species s is a constant in time, as measured along the orbit of a hypothetical particle of species s.

3 PLASMA KINETIC EQUATION

The solutions of the Klimontovich equation would contain all of the particle orbits and would be far too detailed for any practical purpose. It tells us whether or not a particle with infinite density is to be found at a given point $(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{P}$. What we really want to know is how many particles are likely to be found in a

small volume $\Delta\boldsymbol{x}\Delta\boldsymbol{v}$ of phase space, centered at $(\boldsymbol{x},\boldsymbol{v})$. Thus we are really not interested in the spikey function $N_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t)$, but rather in the smooth function

$$f_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t) = \langle N_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t)\rangle, \tag{15}$$

where $\langle\rangle$ is an *ensemble average over an infinite number of realizations of the plasma*.

The distribution function $f_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t)$ is the number of particles of species s per unit configuration space per unit velocity space. Consider a box of a size much greater than a mean interparticle spacing but much smaller than a Debye length. We can now count the number of particles of species s in the box at time t with velocities in the range $\boldsymbol{v}$ to $\boldsymbol{v}+\Delta\boldsymbol{v}$, divide by a unit volume $\Delta\boldsymbol{x}\Delta\boldsymbol{v}$ and call the result $f_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t)$. This number will of course fluctuate with time but, if there are very many particles in the box, the fluctuations will be tiny.

An equation for the time evolution of $f_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t)$ can be obtained from the Klimontovich equation by ensemble averaging under consideration of

$$N_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t) = f_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t) + \delta N_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t) \tag{16}$$

$$\boldsymbol{E}^{\mathrm{m}}(\boldsymbol{x},\boldsymbol{v},t) = \boldsymbol{E}(\boldsymbol{x},\boldsymbol{v},t) + \delta\boldsymbol{E}(\boldsymbol{x},\boldsymbol{v},t) \tag{17}$$

$$\boldsymbol{B}^{\mathrm{m}}(\boldsymbol{x},\boldsymbol{v},t) = \boldsymbol{B}(\boldsymbol{x},\boldsymbol{v},t) + \delta\boldsymbol{B}(\boldsymbol{x},\boldsymbol{v},t), \tag{18}$$

where $\langle\boldsymbol{B}^{\mathrm{m}}\rangle = \boldsymbol{B}$ and $\langle\boldsymbol{E}^{\mathrm{m}}\rangle = \boldsymbol{E}$, and $\langle\delta N_{\mathrm{s}}\rangle = 0$, $\langle\delta\boldsymbol{E}\rangle = \langle\delta\boldsymbol{B}\rangle = \boldsymbol{0}$. The **plasma kinetic equation** is

$$\frac{\partial f_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f_{\mathrm{s}} + \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{v}}f_{\mathrm{s}}$$
$$= -\frac{q_{\mathrm{s}}}{m_{\mathrm{s}}}\langle(\delta\boldsymbol{E} + \boldsymbol{v}/c\times\delta\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}}\delta N_{\mathrm{s}}\rangle. \tag{19}$$

The LHS varies smoothly in $\mathbb{P}$, while the RHS is the ensemble average of products of very spikey quantities. Thus, the LHS is insensitive to the discrete-particle nature of the plasma, while the RHS is very sensitive to the discrete-particle nature of the plasma, which gives rise to collisional effects. The LHS represents collective effects, while the RHS represents collisional effects. The ratio of the importance of collisional effects to the importance of collective effects is $\mathcal{O}(\Lambda^{-1})$, which is a very small number. For many phenomena in a plasma the RHS might be neglected. Letting go $n_0\to\infty, m_e\to 0, e\to 0$ simultaneously, which is called *pulverization procedure*, while $n_0 e = \mathrm{const}, e/m_e = \mathrm{const}$, the LHS becomes infinite and the RHS constant. Thus, the relative importance of the RHS vanishes and we are left with the **Vlasov equation**,

$$\frac{\partial f_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f_{\mathrm{s}} + \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{v}}f_{\mathrm{s}} = 0, \tag{20}$$

sometimes also called **collisionless Boltzmann equation**. $\boldsymbol{E}$ and $\boldsymbol{B}$ are the ensemble averaged fields that satisfy the ensemble averaged version of Maxwell's equations.

4 LIOUVILLE EQUATION

Like the Klimontovich equation, the Liouville equation is exact. It provides a starting point for an approximate form of the RHS of the plasma kinetic equation, which tells us how the distribution function changes in time due to collisions. The Klimontovich equation describes the behaviour of individual particles. By contrast, the Liouville equation describes the behaviour of systems.

Consider a system of two particles. We introduce a set of coordinate axes for each particle, $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ for particle 1 and $(\boldsymbol{x}_2, \boldsymbol{v}_2)$ for particle 2. There is one system in this 12-D phase space. The density of systems, each having two particles, in this phase space is

$$N(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t) = \delta(\boldsymbol{x}_1 - \boldsymbol{X}_1(t))\delta(\boldsymbol{v}_1 - \boldsymbol{V}_1(t))\delta(\boldsymbol{x}_2 - \boldsymbol{X}_2(t))\delta(\boldsymbol{v}_2 - \boldsymbol{V}_2(t)). \tag{21}$$

Generalizing to a system of N_0 particles we associate a 6-D coordinate system $\mathbb{P}_i$ with coordinates $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ with each particle i and thereby construct a $6N_0$-D phase space $\mathbb{P}$. The density of systems in this phase space is

$$N(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, \ldots, \boldsymbol{x}_{N_0}, \boldsymbol{v}_{N_0}, t) = \prod_{i=1}^{N_0} \delta(\boldsymbol{x}_i - \boldsymbol{X}_i(t))\delta(\boldsymbol{v}_i - \boldsymbol{V}_i(t)). \tag{22}$$

As with the Klimontovich equation, the Liouville equation is obtained by taking the time derivative of the appropriate density. Because the density of systems is the product of $6N_0$ terms, its time derivative involves the sum of $6N_0$ terms. The time derivative is

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \boldsymbol{V}_i(t) \cdot \nabla_{\boldsymbol{x}_i} \prod_{j=1}^{N_0} \delta(\boldsymbol{x}_j - \boldsymbol{X}_j)\delta(\boldsymbol{v}_j - \boldsymbol{V}_j)$$
$$+ \sum_{i=1}^{N_0} \dot{\boldsymbol{V}}_i(t) \cdot \nabla_{\boldsymbol{v}_i} \prod_{j=1}^{N_0} \delta(\boldsymbol{x}_j - \boldsymbol{X}_j)\delta(\boldsymbol{v}_j - \boldsymbol{V}_j) = 0 \tag{23}$$

which after short rearrangement gives the **Liouville equation**

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \boldsymbol{v}_i \cdot \nabla_{\boldsymbol{x}_i} N + \sum_{i=1}^{N_0} \dot{\boldsymbol{V}}_i(t) \cdot \nabla_{\boldsymbol{v}_i} N = 0. \tag{24}$$

It contains all of the exact 6-D orbits of the individual particles in a single system orbit in $6N_0$-D space. Again, the equation has the form of a convective time derivative in the $6N_0$-D phase space,

$$\frac{\mathrm{D}}{\mathrm{D}t} N(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, \ldots, \boldsymbol{x}_{N_0}, \boldsymbol{v}_{N_0}, t) = 0. \tag{25}$$

The density of systems, taken along the system orbit is zero – the density of systems is *incompressible*. It can be put in the form of a *continuity equation*

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \nabla_{\boldsymbol{x}_i} \cdot (\boldsymbol{v}_i N) + \sum_{i=1}^{N_0} \nabla_{\boldsymbol{v}_i} \cdot (\dot{\boldsymbol{V}}_i N) = 0 \tag{26}$$

where it expresses the *conservation of systems* in $6N_0$-D phase space. *As the individual particle of the system move about in 6-D phase space, the system itself moves along a continuous orbit in $6N_0$-D phase space.* Each system in the ensemble moves along an orbit, carrying its piece of probability with it. A large probability for point A at time t_0 implies a large probability for point B at time $t > t_0$, provided that both points lie on the same trajectory in $\mathbb{P}$. We can think of the probability density f_{N_0} as a fluid moving in $6N_0$-D phase space. Since probability is neither created nor destroyed and each element of the probability fluid moves along a continuous orbit, the probability fluid satisfies a continuity equation in $6N_0$-D phase space,

$$\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \boldsymbol{v}_i \cdot \nabla_{\boldsymbol{x}_i} f_{N_0} + \sum_{i=1}^{N_0} \dot{\boldsymbol{V}}_i \cdot \nabla_{\boldsymbol{v}_i} f_{N_0} = 0. \tag{27}$$

Of course, the total probability, i.e. f_{N_0} integrated over all $\mathbb{P}$ gives unity. The probability density f_{N_0} is incompressible also, $\mathrm{D}f_{N_0}/\mathrm{D}t = 0$.

5 BBGKY HIERARCHY

The probability density f_{N_0} represents the joint probability density that particle 1 has coordinates between $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ and $(\boldsymbol{x}_1 + \mathrm{d}\boldsymbol{x}_1, \boldsymbol{v}_1 + \mathrm{d}\boldsymbol{v}_1)$ *and* particle 2 has coordinates between $(\boldsymbol{x}_2, \boldsymbol{v}_2)$ and $(\boldsymbol{x}_2 + \mathrm{d}\boldsymbol{x}_2, \boldsymbol{v}_2 + \mathrm{d}\boldsymbol{v}_2)$ *and* ... *and* particle N_0 has coordinates between $(\boldsymbol{x}_{N_0}, \boldsymbol{v}_{N_0})$ and $(\boldsymbol{x}_{N_0} + \mathrm{d}\boldsymbol{x}_{N_0}, \boldsymbol{v}_{N_0} + \mathrm{d}\boldsymbol{v}_{N_0})$. We may also consider reduced probabiliy distributions

$$f_k(\boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{v}_k, t) = V^k \int \mathrm{d}\boldsymbol{x}_{k+1} \int \mathrm{d}\boldsymbol{v}_{k+1} \ldots \int \mathrm{d}\boldsymbol{x}_{N_0} \int \mathrm{d}\boldsymbol{v}_{N_0} \, f_{N_0} \tag{28}$$

which give the joint probability of particles 1 through k having coordinates between $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ and $(\boldsymbol{x}_1 + \mathrm{d}\boldsymbol{x}_1, \boldsymbol{v}_1 + \mathrm{d}\boldsymbol{v}_1)$ *and*... *and* $(\boldsymbol{x}_k, \boldsymbol{v}_k)$ and $(\boldsymbol{x}_k + \mathrm{d}\boldsymbol{x}_k, \boldsymbol{v}_k + \mathrm{d}\boldsymbol{v}_k)$, irrespective of the coordinates of particles $k+1, k+2, \ldots, N_0$. The factor V^k is a normalization factor. We assume that $f_{N_0} \to 0$ as $x_i, y_i, z_i \to \pm\infty$ and $f_{N_0} \to 0$ as $v_{x_i}, v_{y_i}, v_{z_i} \to \pm\infty$. Furthermore f_{N_0} is symmetric with respect to particle labels. This means, if we set $k = 1$, the function $f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$ is the number of particles per unit real space per unit velocity space and has the same meaning as the function $f_\mathrm{s}(\boldsymbol{x}, \boldsymbol{v}, t)$ of the plasma kinetic equation. From now on we adopt the **Coulomb model**, which ignores the magnetic fields produced by the charged particle motion. In this model, the accelaeration is

$$\dot{\boldsymbol{V}}_i(t) = \sum_{j=1}^{N_0} \boldsymbol{a}_{ij} \tag{29}$$

where

$$\boldsymbol{a}_{ij} = \frac{q_\mathrm{s}^2}{m_\mathrm{s}|\boldsymbol{x}_i - \boldsymbol{x}_j|^3}(\boldsymbol{x}_i - \boldsymbol{x}_j) \tag{30}$$

is the acceleration of particle i due to the Coulomb electric field of particle j, and, of course $\boldsymbol{a}_{ii} = \boldsymbol{0}\,\forall i$. This equation replaces Maxwell's equations and the Lorentz force law! The Liouville equation becomes

$$\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \boldsymbol{v}_i \cdot \nabla_{\boldsymbol{x}_i} f_{N_0} + \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} \boldsymbol{a}_{ij} \cdot \nabla_{\boldsymbol{v}_i} f_{N_0} = 0. \tag{31}$$

Equations for the reduced distributions f_k are obtained by integrating the Liouville equation over all $\boldsymbol{x}_{k+1}\boldsymbol{v}_{k+1},\ldots,\boldsymbol{x}_{N_0}\boldsymbol{v}_{N_0}$. E.g. to obtain the equation for f_{N_0-1} we integrate over all $\boldsymbol{x}_{N_0}\boldsymbol{v}_{N_0}$, obtaining

$$\int \mathrm{d}\boldsymbol{x}_{N_0} \int \mathrm{d}\boldsymbol{v}_{N_0} \frac{\partial f_{N_0}}{\partial t} + \int \mathrm{d}\boldsymbol{x}_{N_0} \int \mathrm{d}\boldsymbol{v}_{N_0} \sum_{i=1}^{N_0} \boldsymbol{v}_i \cdot \nabla_{\boldsymbol{x}_i} f_{N_0}$$

$$+ \int \mathrm{d}\boldsymbol{x}_{N_0} \int \mathrm{d}\boldsymbol{v}_{N_0} \sum_{i=1}^{N_0}\sum_{i=1}^{N_0} \boldsymbol{a}_{ij} \cdot \nabla_{\boldsymbol{v}_i} f_{N_0} = 0. \tag{32}$$

where the first term is

$$\frac{\partial}{\partial t} \int \mathrm{d}\boldsymbol{x}_{N_0} \int \mathrm{d}\boldsymbol{v}_{N_0} f_{N_0} = V^{1-N_0} \frac{\partial}{\partial t} f_{N_0-1}. \tag{33}$$

Simplifying the above equation one gets an equation for f_{N_0-1} which still depends on f_{N_0}. It is exact within the Coulomb model. By the same procedure we can calculate f_{N_0-2}, which also still involves f_{N_0-1}. Comparing the pattern of the emerging equations, the following relation

$$\frac{\partial}{\partial t} f_k + \sum_{i=1}^{k} \boldsymbol{v}_i \cdot \nabla_{\boldsymbol{x}_i} f_k + \sum_{i=1}^{k}\sum_{j=1}^{k} \boldsymbol{a}_{ij} \cdot \nabla_{\boldsymbol{v}_i} f_k$$

$$+ \frac{N_0 - k}{V} \sum_{i=1}^{k} \int \mathrm{d}\boldsymbol{x}_{k+1} \int \mathrm{d}\boldsymbol{v}_{k+1} \, \boldsymbol{a}_{i,k+1} \cdot \nabla_{\boldsymbol{v}_i} f_{k+1} = 0. \tag{34}$$

has been obtained by Bogoliubov, Born, Green, Kirkwood and Yvon. It is the so-called **BBGKY hierarchy**. As it stands, the BBGKY hierarchy is still exact within the Coulomb model. It consists of N_0 coupled integro-differential equations. Progress will come only when we take just the first few equations for $k = 1$, $k = 2$, etc. and then use an approximation to close the set and cut off the dependence on higher-order equations.

The $k = 1$ equation is

$$\partial_t f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t) + \boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} f_1 + \frac{N_0 - 1}{V} \int \mathrm{d}\boldsymbol{x}_2 \int \mathrm{d}\boldsymbol{v}_2 \, \boldsymbol{a}_{12}$$

$$\cdot \nabla_{\boldsymbol{v}_1} f_2(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t) = 0. \tag{35}$$

This is coupled to the $k = 2$ equation via f_2. One way to proceed is to find some approximation for f_2 in terms of f_1. $f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t)\,\mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{v}_1$ is the probability that a

given particle finds itself in the region of phase space between $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ and $(\boldsymbol{x}_1 + \mathrm{d}\boldsymbol{x}_1, \boldsymbol{v}_1 + \mathrm{d}\boldsymbol{v}_1)$. The function f_2 is the ensemble averaged number of particles per unit $\boldsymbol{x}_1$ real space per unit $\boldsymbol{x}_2$ real space per unit $\boldsymbol{v}_1$ velocity space per unit $\boldsymbol{v}_2$ velocity space. $f_2(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t)$ is proportional to the joint probability that particle 1 finds itself at $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ *and* particle 2 finds itself at $(\boldsymbol{x}_2, \boldsymbol{v}_2)$. If the joint probability would be the probability of two *statistically independent* quantities, then simply $f_2(1,2) = f_1(1)f_1(2)$[1] Yet, if these quantities are not completely uncorrelated, we have to consider a correlation term g as follows,

$$f_2(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t) = f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t)f_1(\boldsymbol{x}_2, \boldsymbol{v}_2, t) + g(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t). \qquad (36)$$

This is called the **Mayer cluster expansion**. Substitution into the equation for f_1 yields

$$\partial_t f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t) + \boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} f_1 + n_0 \int \mathrm{d}\boldsymbol{x}_2 \int \mathrm{d}\boldsymbol{v}_2\, \boldsymbol{a}_{12} \cdot \nabla_{\boldsymbol{v}_1}(f_1(1)f_1(2) + g(1,2,t)) = 0 \qquad (37)$$

thereby also replacing $(N_0 - 1)/V \approx n_0$ because $N_0 \gg 1$.

Let us just once assume that $g = 0$. That is, the particles in the plasma behave as if they were completely independent of the particular positions and velocities of the other particles. This would be true by performing the *pulverization procudure*: $n_0 \to \infty$, $m_e \to 0$, $\Lambda \to \infty$, $n_0 e = \mathrm{const}$, $e/m_e = \mathrm{const}$. Then each particle would have zero charge and its presence would not affect any other particle. Collective effects would still happen, as these involve only f_1 and not g. Considering that the ensemble averaged acceleration experienced by particle 1 due to all other particles is

$$\boldsymbol{a}(\boldsymbol{x}, t) = n_0 \int \mathrm{d}\boldsymbol{x}_2 \int \mathrm{d}\boldsymbol{v}_2\, \boldsymbol{a}_{12} f_1(\boldsymbol{x}_2, \boldsymbol{v}_2, t) \qquad (38)$$

we are then left with

$$\partial_t f_1 + \boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} f_1 + \boldsymbol{a} \cdot \nabla_{\boldsymbol{v}_1} f_1 = 0, \qquad (39)$$

which, again is the Vlasov equation. It does not include collisional effects that are represented by the two-particle correlation function g.

Yet, we would like to have at least an approximate equation that does include collisional effects and that, therefore, predicts the temporal evolution of f_1 due to collisions. We must therefore return to the exact $k = 1$ and $k = 2$ equations and find some method to evalueate $g = f_2(1,2) - f_1(1)f_1(2)$. Setting $k = 2$ we have

$$\partial_t f_2 + (\boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} + \boldsymbol{v}_2 \cdot \nabla_{\boldsymbol{x}_2})f_2 + (\boldsymbol{a}_{12} \cdot \nabla_{\boldsymbol{v}_1} + \boldsymbol{a}_{21} \cdot \nabla_{\boldsymbol{v}_2})f_2$$
$$+ n_0 \int \mathrm{d}\boldsymbol{x}_3 \int \mathrm{d}\boldsymbol{v}_3 (\boldsymbol{a}_{13} \cdot \nabla_{\boldsymbol{v}_1} + \boldsymbol{a}_{23} \cdot \nabla_{\boldsymbol{v}_2})f_3 = 0. \qquad (40)$$

---

[1]Here we introduced the shorthand notation $(1,2) = (\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2)$

The next order in the Mayer cluster expansion is

$$f_3(123) = f_1(1)f_2(2)f_3(3) + f_1(1)g(23) + f_1(2)g(13) + f_1(3)g(12) + h(123). \quad (41)$$

We neglect h, i.e. three-particle correlations or three-particle collisions. These are of higher order in Λ^{-1}. The resulting set of equations constitute two equations in two unknowns f_1 and g. Thus we have truncated the BBGKY hierarchy while retaining the effects of collisions to a good approximation. Substitution of f_3 yields two equations, one for the time evolution of $g(12)$ and one for the time evolution of $f_1(1)$:

$$
\begin{aligned}
\dot{g}(12) &+ (\boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} + \boldsymbol{v}_2 \cdot \nabla_{\boldsymbol{x}_2})g(12) \\
&= -(\boldsymbol{a}_{12} \cdot \nabla_{\boldsymbol{v}_1} + \boldsymbol{a}_{21} \cdot \nabla_{\boldsymbol{v}_2})[f_1(1)f_1(2) + g(12)] \\
&\quad - \left\{ n_0 \int \mathrm{d}\boldsymbol{x}_3 \int \mathrm{d}\boldsymbol{v}_3\, \boldsymbol{a}_{13} \cdot \nabla_{\boldsymbol{v}_1}[f_1(1)g(23) + f_1(3)g(12)] + (1 \leftrightarrow 2) \right\} \quad (42)
\end{aligned}
$$

$$\dot{f}_1(1) + \boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} f_1 + \boldsymbol{a} \cdot \nabla_{\boldsymbol{v}_1} f_1 = -n_0 \int \mathrm{d}\boldsymbol{x}_2 \int \mathrm{d}\boldsymbol{v}_2\, \boldsymbol{a}_{12} \cdot \nabla_{\boldsymbol{v}_1} g(12) \quad (43)$$

The simplifications so far have been:

- $1 \ll N_0 \Rightarrow (N_0 - 1)/V \approx n_0$

- neglect of h, i.e. three-particle collisions, though in reality each particle is interacting with Λ particles simultaneously

- Coulomb model

So far these approximations have been extremely good ones – by contrast, the simplifications needed to convert these two equations into managable form are sometimes quite drastic and less justifiable as we will see in the next section.

6   BOGOLIUBOV'S HYPOTHESIS

Further assumptions are needed to simplify these two equations. Consider a spatially homogeneous ensemble of plasmas. Then, any ensemble averaged function of two spatial variables can only be a function of the difference between those variables and any function of a spatial variable must be independent of that variable. These assumptions simplifies the set of equations considerably to

$$\partial_t f_1(\boldsymbol{v}_1, t) = -n_0 \int \mathrm{d}\boldsymbol{x}_2 \int \mathrm{d}\boldsymbol{v}_2\, \boldsymbol{a}_{12} \cdot \nabla_{\boldsymbol{v}_1} g(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t) \quad (44)$$

and

$$\frac{\partial g(12)}{\partial t} + V_1 g + V_2 g = S \quad (45)$$

by application of the pulverization procedure and comparision of the relative importance of terms w.r.t. to $\mathcal{O}(\Lambda^{-1})$. The operator V_1 acting on g and the source function S are defined as

$$V_1 g(12) = \boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} g(12) + [n_0 \int \mathrm{d}3\, \boldsymbol{a}_{13} g(23)] \cdot \nabla_{\boldsymbol{v}_1} f_1(1). \qquad (46)$$

and

$$S(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) = -(\boldsymbol{a}_{12} \cdot \nabla_{\boldsymbol{v}_1} + \boldsymbol{a}_{21} \cdot \nabla_{\boldsymbol{v}_2}) f_1(1) f_1(2), \qquad (47)$$

respectively. The important physical situation to which this discussion applies is as follows: imagine a beam of electrons incident on a Maxwellian electron plasma. The beam of electrons represented by the bump at large positive v will eventually produce a new Maxwellian at a higher temperature. This process is relevant e.g. for ion ohmic heating of a tokamak.
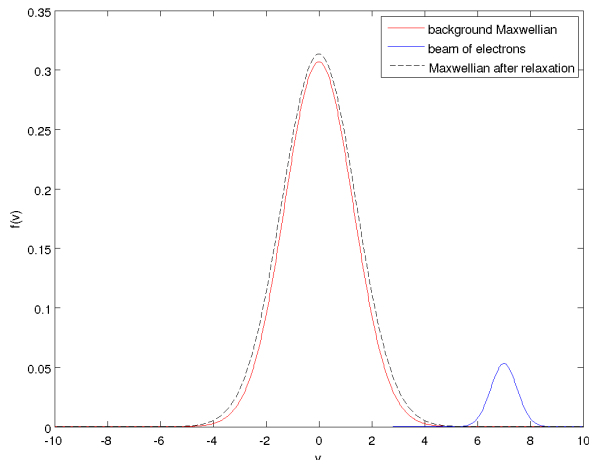

Figure 1: Beam of electrons relaxing to Maxwellian for $t \to \infty$

Bogoliubov's hypothesis assumes that the two-point correlation function g relaxes on a time scale very short compared to the time scale on which f_1 relaxes. Mathematically this assumption is incorporated by ignoring the time dependence of $f_1(\boldsymbol{v}_1, t)$ and $f_1(\boldsymbol{v}_2, t)$ in the source function S. The resultant equation is linear for g with a constant source function on the right. This linear equation for $g(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t \to \infty)$ can be solved on the short time scale ($t \to \infty$). The solution for g is then substituted on the equation for the time evolution of f_1. By doing so we finally truncate the BBGKY hierarchy and express the entire plasma kinetic equation in terms of one unknown function $f_1(\boldsymbol{v}_1, t)$. The solution is obtained by Fourier transformation in space, Laplace

transformation in time, and their inverses. It is stated here without derivation:

$$\frac{\partial f(\boldsymbol{v}, t)}{\partial t} = -\frac{8\pi^4 n_0}{m_e^2} \nabla_{\boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{v}' \, \boldsymbol{k} \otimes \boldsymbol{k} \frac{\varphi^2(k)}{|\epsilon(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^2}$$
$$\cdot \, \delta(\boldsymbol{k} \cdot (\boldsymbol{v} - \boldsymbol{v}'))[f(\boldsymbol{v})\nabla_{\boldsymbol{v}'} f(\boldsymbol{v}') - f(\boldsymbol{v}')\nabla_{\boldsymbol{v}} f(\boldsymbol{v})], \qquad (48)$$

where $\varphi(k)$ is the Fourier transform of the Coulomb potential $\varphi(x) = e^2/|\boldsymbol{x}|$, i.e. $\varphi(k) = e^2/(2\pi^2 k^2)$. This is the **Lenard-Balescu equation**. We also introduced the *dielectric function*,

$$\epsilon(\boldsymbol{k}, \omega) = 1 + \frac{\omega_e^2}{k^2} \int \mathrm{d}\boldsymbol{v} \frac{\boldsymbol{k} \cdot \nabla_{\boldsymbol{v}} f(\boldsymbol{v})}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}}, \qquad (49)$$

which represents the plasma shielding of the field of a test charge. Due to the assumptions made earlier, the Lenard Balescu equation is applicable to situations such as the collisional relaxation of a beam in a plasma, but is not applicable in general to any phenomena that involve high frequencies.

At large k the integral diverges like $\int \mathrm{d}k/k \sim \ln k$, we find a logarithmic divergence at large k, or small distances. The derivation of the Lenard-Balescu equation is based on the assumption that in the expression

$$f_2(12) = f_1(1)f_1(2) + g(12) \qquad (50)$$

we have $|g| \ll |f_1 f_1|$. However, this assumption is not always valid. It is not possible for two electrons to get very close to each other. Therefore, we must have $f_2 \to 0$ as $\boldsymbol{x}_1 \to \boldsymbol{x}_2$, which implies $g = -f_1 f_1$. Thus, for small values of $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$ (i.e. for large k), it is not correct to assume $|g| \ll |f_1 f_1|$. In practice, since the divergence is logarithmic, we can simply cut off the integral at some upper limit wave number corresponding to some lower limit spatial scale. For this purpose, the impact parameter p_0 for large angle collisions is a reasonable choice. The error that is commited thereby contributes only logarithmically. As $t \to \infty$, any f approaches a Maxwellian.

We can put the Lenard-Balescu equation in more compact form by introduction of the tensor $\mathbf{Q}$

$$\mathbf{Q}(\boldsymbol{v}, \boldsymbol{v}') = -\frac{8\pi^4 n_0}{m_e^2} \int \mathrm{d}\boldsymbol{k} \, \boldsymbol{k} \otimes \boldsymbol{k} \frac{\varphi^2(k)}{|\epsilon(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^2} \delta(\boldsymbol{k} \cdot (\boldsymbol{v} - \boldsymbol{v}')) \qquad (51)$$

as follows:

$$\frac{\partial f(\boldsymbol{v}, t)}{\partial t} = -\nabla_{\boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{v}' \mathbf{Q}(\boldsymbol{v}, \boldsymbol{v}') \cdot (\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}'}) f(\boldsymbol{v}) f(\boldsymbol{v}'). \qquad (52)$$

We can orient the $\hat{k}_1$ axis in the $\boldsymbol{v} - \boldsymbol{v}'$ direction from which follows that $Q_{ij} = 0$ for all i, j apart from the terms

$$Q_{22}(\boldsymbol{v}, \boldsymbol{v}') = Q_{33}(\boldsymbol{v}, \boldsymbol{v}') = -\frac{2\pi n_0 e^4}{m_e^2 |\boldsymbol{v} - \boldsymbol{v}'|} \ln \Lambda \qquad (53)$$

11

which can be conveniently expressed by

$$\mathbf{Q}(\boldsymbol{v}, \boldsymbol{v}') = -\frac{2\pi n_0^4 \ln \Lambda}{m_\mathrm{e}^2} \frac{g^2 \mathbf{I} - \boldsymbol{g} \otimes \boldsymbol{g}}{g^3}. \tag{54}$$

We can further transform this equation to

$$\frac{\partial f(\boldsymbol{v}, t)}{\partial t} = -\nabla_{\boldsymbol{v}} \cdot [\boldsymbol{A} f(\boldsymbol{v})] + \frac{1}{2} \nabla_{\boldsymbol{v}} \otimes \nabla_{\boldsymbol{v}} : [\mathbf{B} f(\boldsymbol{v})] \tag{55}$$

which is the standard form[2] of a **Fokker-Planck equation** with $\boldsymbol{A}$ being the *coefficient of dynamic friction*

$$\boldsymbol{A}(\boldsymbol{v}, t) = \frac{8\pi n_0 e^4 \ln \Lambda}{m_e^2} \nabla_{\boldsymbol{v}} \int \mathrm{d}\boldsymbol{v}' \frac{f(\boldsymbol{v}', t)}{|\boldsymbol{v} - \boldsymbol{v}'|} \tag{56}$$

and $\mathbf{B}$, the *diffusion coefficient*

$$\mathbf{B}(\boldsymbol{v}, t) = \frac{4\pi n_0 e^4 \ln \Lambda}{m_e^2} \nabla_{\boldsymbol{v}} \otimes \nabla_{\boldsymbol{v}} \int \mathrm{d}\boldsymbol{v} |\boldsymbol{v} - \boldsymbol{v}'| f(\boldsymbol{v}', t). \tag{57}$$

The coefficient of dynamic friction $\boldsymbol{A}$ represents the slowing down of a typical particle due to many small angle collisions. The diffusion coefficient $\mathbf{B}$ represents the increase of a typical particle's velocity in the direction perpendicular to its instantaneous velocity due to many small angle collisions. Thus the two terms on the RHS of the Fokker-Planck equation tend to balance each other.

A simpler, less accurate form of the Fokker-Planck equation is given by the *Krook model*,

$$\frac{\partial f}{\partial t} = -\nu(f - f_0) \tag{58}$$

where ν is a collision frequency, and f_0 is the appropriate Maxwellian distribution. This equation is also called the BGK equation, after Bhatnagar, Gross and Krook.

7 Appendix – THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation is a very general equation in physics; it describes not only Brownian particles, but any phenomenon that in some approximate sense can be thought of as a **Markov process**. A Markov process is one whose value at the next measuring time depends only on its value at the present measuring time, and not on any previous measuring time. Thus, if $x(t)$ is the random process, and $x_n = x(t_n)$ with $t_n > t_{n-1} > \ldots > t_1 > t_0$, a Markov process has a probability density such that

$$\rho(x_n | x_{n-1} x_{n-2} \ldots x_1 x_0) = \rho(x_n | x_{n-1}) \tag{59}$$

---

[2]The double dot product of two tensors is defined by $\mathbf{P} : \mathbf{R} = \sum_{i,j} P^{ij} R_{ji}$ or also $\boldsymbol{X}\boldsymbol{A} : \boldsymbol{S}\boldsymbol{T} = \boldsymbol{A} \cdot \boldsymbol{S}\boldsymbol{T} \cdot \boldsymbol{X} = (\boldsymbol{A} \cdot \boldsymbol{S})(\boldsymbol{T} \cdot \boldsymbol{X})$

where the notion $\rho(a|b)$ means the probability density of a given that b was true. Thus for a Markov process, the probability that $x_n = 5$ depends only on what the value of x_{n-1} was; it does not depend on what the values of x_{n-2}, x_{n-3}, etc. were. An example for a discrete Markov process is given by flipping a coin. Here $\rho(x_n)$ does not depend on x_{n-1}, much less on x_{n-2}, x_{n-3}, etc.

To give an example of a continous Markov process is more difficult, because continuous Markov processes cannot exist in nature. Any random function in nature can be drawn as a smooth curve. Now it appears that x_{n+1} not only depends on x_n but also on x_{n-1}. That is, x_{n+1} not only depends on x_n, but also on the derivative of the function $dx(t)/dt|_{t=t_0}$, which can be written

$$\left.\frac{dx(t)}{dt}\right|_{t=t_0} = \frac{x_n - x_{n-1}}{\Delta t}. \tag{60}$$

Thus, such a function is not a Markov process. In fact, no function that is a continuous curve and, therefore, no physical function, can be a Markov process.

This does not mean that Markov processes cannot be a good approximation to a physical process. Consider the velocity function of a Brownian particle. It consists of rapid fluctuations due to each molecular collision, together with a slowing down or net friction force. Thus, on the time scale of molecular collisions, the process is not Markovian. However, on the much longer time scale of many collision times, the situation is very nearly Markovian. The Brownian particle is performing a random walk in velocity space, and soon forgets the details of its orbit near $t = 0$. For a Markov process

$$\rho(x_n, x_{n-1}, x_{n-2}, \ldots, x_0) = \rho(x_n|x_{n-1})\rho(x_{n-1}|x_{n-2})\ldots\rho(x_2|x_1)\rho(x_1|x_0)\rho(x_0) \tag{61}$$

and also

$$\rho(x_2|x_0) = \int dx_1 \, \rho(x_2|x_1)\rho(x_1|x_0) \tag{62}$$

which is the famous *Chapman-Kolmogorov equation*. If we assume that all of the important physics happens for small Δx and take the limit as $\Delta t \to 0$, i.e. letting Δt become very small, much smaller than any macroscopic time scale, we arrive at the *Fokker-Planck equation*

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}[D^{(1)}(x,t)\rho(x,t)] + \frac{\partial^2}{\partial x^2}[D^{(2)}(x,t)\rho(x,t)]. \tag{63}$$

For Brownian motion, the random variable x is replaced by the particle velocity $v(t)$. This we can compare to the *Lenard-Balescu equation* in the form of a Fokker-Planck equation,

$$\frac{\partial f(\boldsymbol{v}_1, t)}{\partial t} = -\nabla_{\boldsymbol{v}_1} \cdot (\boldsymbol{A}f) + \frac{1}{2}\nabla_{\boldsymbol{v}_1}\nabla_{\boldsymbol{v}_1} : (\mathbf{B}f). \tag{64}$$

Because we assumed $g(1,2) \ll f_1(1)f_1(2)$ in the derivation of the Lenard Balescu equation, we have effectively limited ourselves to small angle two-body collisions. The quantity $f(\boldsymbol{v}_1, t)$ may be thought of as the probability density of particles

in velocity space. Thus, $f(\boldsymbol{v}_1, t)$ is changing slowly on the time scale for a two-body collision. The coefficient $\boldsymbol{A}$ represents the slowing down of a particle due to many small angle Coulomb collisions. Likewise, the coefficient $\boldsymbol{\mathsf{B}}$ represents the diffusion of the plasma particles in velocity space due to many small angle collisions. In the steady state, a typical particle is suffering dynamic friction plus diffusion; the net effect is to produce a Maxwellian.

References

[1] Dwight R. Nicholson: *Introduction to Plasma Theory*, John Wiley & Sons