NUMERISCHE METHODEN IN DER PHYSIK
Fourth Exercise WS 2013/2014 [MATLAB]

Numerical evaluation of initial value problems using MATLAB routines

- In this exercise we want to use the computational routines provided by Matlab to numerically solve the initial value problems of the following examples:
  - Satellite example from the lecture notes
    Data for comparisons can be found in the file *exakte_ellipse*.
  - Trajectories of charged particles in a magnetic field
  - Oscillation behaviour of van der Pol oscillators
- The most important program of the matlab library in this case is *ode45*.

  Its name can be easily explained: ODE stands for *ordinary differential equation* and 45 signifies that it's using a combination of two *Runge-Kutta* algorithms of orders 4 and 5. These Runge-Kutta algorithms are closely related to the *Runge-Kutta-Fehlberg* method introduced in the lecture notes in Sec. 8.6 (German version).
- You can get a detailed view of all available functions and programs for solving initial value problems in Matlab by browsing the HELP section of Matlab. There you can also read up the exact way of how to use a routine with a description of all input and output parameters.

Proper use of the Matlab function ode45:

Here we want to provide a very short description on how to use *ode45* properly. A very basic option would be:

```
% Matlab example program 'ode_main.m':

% Defining the interval in which the differential equation
% has to be solved (for initial value problems this is very
% often a time interval).
  time=[tstart tend];

% Creating a vector with all initial values for the functions that
% solve the set of diff. eq.
  y0=[y1_0 y2_0 ....];

% Using the function 'odeset', a positive value for the relative accuracy
% is set:
  TOL= ...;   % z.B.: 0.0001
  options=odeset('RelTol',TOL);

% Call of the program 'ode45':
% Here we expect the existence of a Matlab file called 'odefunc.m',
% in which the functions f of the set of diff. eq. are defined:
  [timeout,y] = ode45('odefunc',time,y0,options);

% The output then is:
%   vector 'timeout' with the time arguments;
%   matrix 'y':
%   first to nth column: values of the functions  y1(t)  y2(t) ... yn(t)
  .
  .
  .


% Matlab example programm 'odefunc.m':
%
% This program defines the functions f
% (the right hand side of the set of diff. eq.)
% This function corresponds to the algorithm 'DERIVS'
% described in the lecture notes.

  function f = odefunc(t,y)

  f=zeros(n,1);  % a column vector.  n is the order of the system

  f(1)= ... ;    % function  f1(t,y1,y2,...)
  f(2)= ... ;    % function  f2(t,y1,y2,...)
   .
   .
```

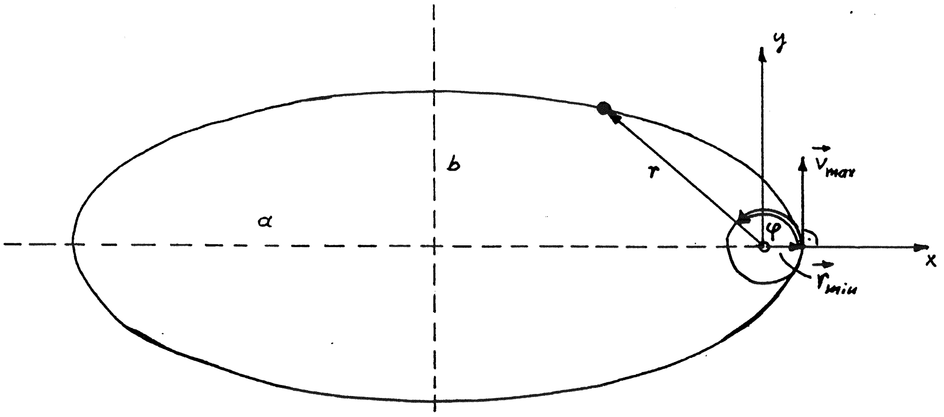

Figure 1: Trajectory of an earth satellite

1. Exercise: Satellite example from the lecture

Test the performance of the Matlab program *ode*45 with the satellite example from the lecture (see German lecture notes starting on page 245).

Defining the problem:

A satellite is set into motion with a velocity v_{max}, tangential to the surface of the earth. We want to calculate its trajectory considering the following constraints and approximations:

- $v_{max} <$ escape velocity.
- The earth shall be approximated as an ideal sphere with r_{min} and homogenous density.
- The influence of the earth's atmosphere is neglected.
- The influence of all other celestial bodies is neglected.

With these constraints, the *equation of motion* of the satellite reads

$$\ddot{\mathbf{r}} = -\frac{\gamma M}{r^3}\mathbf{r}\,, \tag{1}$$

where **r** denotes the vector originating from the earth's center and ending at the satellite's momentary position.

Equation (1) leads in the end to a set of two differential equations of second order for the distance r and the rotation angle φ:

$$\begin{aligned} \ddot{r} &= r\dot{\varphi}^2 - \frac{\gamma M}{r^2} \\ \ddot{\varphi} &= -\frac{2\dot{r}\dot{\varphi}}{r} \end{aligned} \tag{2}$$

Substituting

$$r \to y_1 \quad \varphi \to y_2 \quad \dot{r} \to y_3 \quad \dot{\varphi} \to y_4 \quad ,$$

one gets a set of 4 differential equations of first order:

$$\begin{aligned} \dot{y}_1 &= y_3 & y_1(t=0) &= r_{min} \\ \dot{y}_2 &= y_4 & y_2(t=0) &= 0 \\ \dot{y}_3 &= y_1 y_4^2 - \frac{\gamma M}{y_1^2} & y_3(t=0) &= 0 \\ \dot{y}_4 &= -\frac{2y_3y_4}{y_1} & y_4(t=0) &= \frac{v_{max}}{r_{min}} \end{aligned} \tag{3}$$

Constants and initial values: M and γ are the earth's mass and the gravitational constant

$$M = 5.977 \cdot 10^{24} \text{ kg} \qquad \text{and} \qquad \gamma = 6.67 \cdot 10^{-11} \text{ m}^3/(\text{kg s}^2)\,.$$

The satellite starts at the earth's surface ($r_{min} = 6.37 \cdot 10^6$ m) with a velocity of $v_{max} = 10.4 \cdot 10^3$ m/s.

We want to express all lengths in units of r_{min}, all velocities in units of v_{max} and all times in units of the period of revolution $T = 35705.83$.

In these units the constant $\alpha = \gamma M$ has the value 1966.39, and the intial values for $y_1(0)$ and $y_4(0)$ are 1.0 and 58.29527 respectively.

According to eq. (3) the definition of the functions reads

```
function dy = f_satellit(t,y)

global alpha
dy = zeros(4,1);  % a column vector

dy(1) = y(3);
dy(2) = y(4);
dy(3) = y(1)*y(4)^2 - alpha/(y(1)^2); % alpha GLOBAL = 1966.390
dy(4) = -2*y(3)*y(4)/y(1);
```

Tasks:

- Solve the set of equations for a time interval from 0 to 5 revolutions of the satellite four times, each time with different tolerance values

  ```
  TOL(rel) = 0.025    0.01    0.001    0.0001
  ```

- Present your results in four seperate figures depicting the calculated trajectory in xy space. Compare these trajectories to the exact, analytically calculated elliptical trajectory. You can find the exact data in the file

  ```
  exakte_ellipse
  ```

2. Exercise: Trajectory in an inhomogeneous magnetic field

Theoretical background

A particle with mass m and electrical charge q, which moves in a magnetic field (with magnetic flux density $\mathbf{B}(\mathbf{r}, t)$) with a velocity of $\mathbf{v}(t)$ is subjected to a **Lorentz force**

$$\mathbf{F}_L = q(\mathbf{v} \times \mathbf{B}) \quad . \tag{4}$$

To calculate the **trajectory** of the particle we start with Newton's second law

$$m\ddot{\mathbf{r}}(t) = q\left[\dot{\mathbf{r}}(t) \times \mathbf{B}(\mathbf{r}, t)\right] . \tag{5}$$

This vectorial differential equation of second order corresponds to the set of differential equations

$$\begin{aligned} \ddot{x} &= \frac{q}{m}(\dot{y}B_z - \dot{z}B_y) \\ \ddot{y} &= \frac{q}{m}(\dot{z}B_x - \dot{x}B_z) \\ \ddot{z} &= \frac{q}{m}(\dot{x}B_y - \dot{y}B_x) \end{aligned} \tag{6}$$

Taking the **initial values** for the time $t = 0$ into account

$$\begin{array}{ccc} x(t=0) = x_o & y(t=0) = y_o & z(t=0) = z_o \\ \dot{x}(t=0) = v_{xo} & \dot{y}(t=0) = v_{yo} & \dot{z}(t=0) = v_{zo} \end{array} \tag{7}$$

we are left with a uniquely defined **initial value problem**. The solutions $x(t)$, $y(t)$ and $z(t)$ are the spatial components of the trajectory parametrised by t.

The space- and time-dependent vector of the magnetic field (more precisely of the magnetic flux density) is of the form

$$\mathbf{B}(\mathbf{r}, t) = B_0 \begin{pmatrix} f_1(\mathbf{r}, t) \\ f_2(\mathbf{r}, t) \\ f_3(\mathbf{r}, t) \end{pmatrix} \tag{8}$$

Putting this vector in equation (6) results in a common factor in all of the equations called the angular frequency qB_0/m, which in our calculations is set to the value

$$\frac{qB_0}{m} \equiv \sigma = \pi \ .$$

Your task now is to write a program that can solve numerically the initial value problem described above for arbitrary values of σ, x_o, v_{xo}, y_o, v_{yo}, z_o and v_{zo} as well as for arbitrary functions $f_i(\mathbf{r}, t)$ by using the Matlab function *ode*45.

Testing the program

If the magnetic field is constant in time as well as space and points exactly in z-direction, i.e.

$$f_1(\mathbf{r},t) = 0.0 \qquad f_2(\mathbf{r},t) = 0.0 \qquad f_3(\mathbf{r},t) = 1.0\,,$$

we can solve the problem analytically and obtain the following solution

$$x(t) = -\frac{v_{yo}}{\sigma}\cos(\sigma t) + \frac{v_{xo}}{\sigma}\sin(\sigma t) + x_o + \frac{v_{yo}}{\sigma} \tag{9}$$

$$y(t) = \frac{v_{yo}}{\sigma}\sin(\sigma t) + \frac{v_{xo}}{\sigma}\cos(\sigma t) + y_o - \frac{v_{xo}}{\sigma} \tag{10}$$

$$z(t) = z_o + v_{zo}t\ . \tag{11}$$

In this case, the trajectory of the particle is a helix with the symmetry axis in z-direction. Projecting the helix onto the xy plane one gets a circle with center coordinates $x_o + v_{yo}/\sigma$ and $y_o - v_{xo}/\sigma$ and a radius of $\sqrt{v_{xo}^2 + v_{yo}^2}/\sigma$. In z-direction we have a uniform motion with velocity v_{zo} independent of the magnetic field.

Use these considerations to test your program, i.e. compare your results for $0 \leq t \leq 10$ s with the analytic results from equations (9-11) for following initial values:

$$x_o = 0 \qquad y_o = 0 \qquad z_o = 0 \qquad v_{xo} = 0 \qquad v_{yo} = 2\pi \qquad v_{zo} = 0.25$$

Tolerance TOL(rel) for all calculations $= 10^{-5}$.

Graphic representation:

1. Present the three functions $x(t)$, $y(t)$ and $z(t)$ in one figure (see fig. 2).
2. Show the calculated trajectory in 3D-space, as for example shown in fig. 3.

   You can make use of the very convenient plotting function *plot3(vec-x,vec-y,vec-z)* for these kinds of trajectories $[x(t), y(t), z(t)]$.

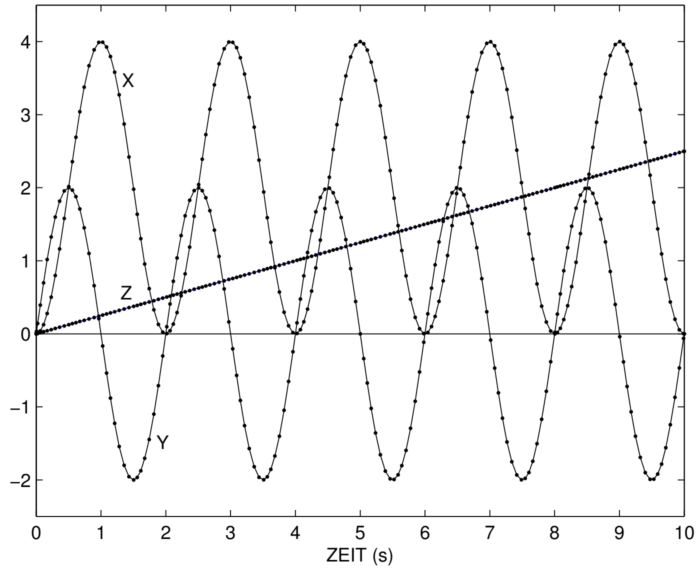

Figure 2: The functions $x(t)$, $y(t)$ and $z(t)$ in the test case.

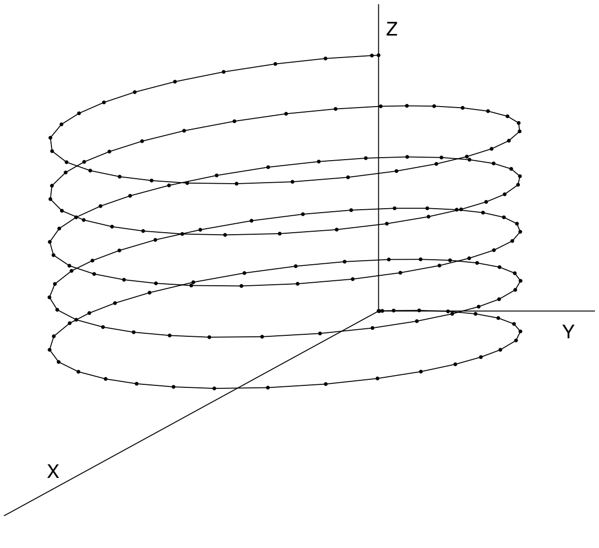

Figure 3: Trajectory of the charged particle for the test example.

Application example

Use the following inhomogeneous magnetic fields:

Example: Magnetic field, inhomogeneous in time:

Calculate the trajectories for the time interval $0 \le t \le 10$ s and the initial values

$$x_o = 0 \qquad y_o = 0 \qquad z_o = 0$$

$$v_{xo} = 0 \qquad v_{yo} = 1.5 \qquad v_{zo} = 2\pi$$

(a) for the magnetic field

$$f_1(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\gamma t} \qquad f_2(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\gamma t} \qquad f_3(\mathbf{r}, t) = 0$$

with

$$\gamma = 0.5\,,$$

(b) for the magnetic field

$$f_1(\mathbf{r}, t) = \sqrt{2}\,\mathrm{e}^{\gamma t} \qquad f_2(\mathbf{r}, t) = \sqrt{2}\,\mathrm{e}^{\gamma t} \qquad f_3(\mathbf{r}, t) = 0$$

with

$$\gamma = -0.25\,.$$

Example: "Tilted" magnetic field:

Calculate the trajectories for the time interval $0 \leq t \leq 20$ s and the initial values

$$x_o = 0 \qquad y_o = 0 \qquad z_o = 0$$

$$v_{xo} = 0 \qquad v_{yo} = 2\pi \qquad v_{zo} = 0.25$$

In the time interval $0 \leq t \leq t_a$ the magnetic field is considered constant and to point in z-direction

$$\mathbf{f}(\mathbf{r}, t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

In the time interval $t_a \leq t \leq t_b$ the vector $\mathbf{f}$ rotates *linearly in time* towards the x-direction (see fig. 4).

For times $t > t_b$ the magnetic field is constant and points in x-direction:

$$\mathbf{f}(\mathbf{r}, t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

Time parameter: (a) $t_a = 8.5$ s $\quad t_b = 9.5$ s
(b) $t_a = 9.0$ s $\quad t_b = 10.0$ s

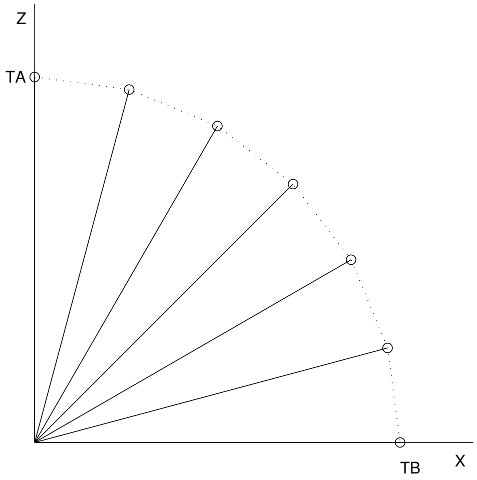

Figure 4: Tilting the magnetic field

3. Exercise: van der Pol oscillator

Theoretical background

We want to study the properties of the so-called *van der Pol* oscillator. This kind of oscillators appears for example in an electromagnetic transmitter, whose losses through the resistance are compensated by backcoupling to a control grid of a triode.
The differential equation describing this case can in principle be *only solved numerically.* There are, however, analytic approximations for the *attractor curve*, as will be described later.

The van der Pol oscillator is described by the following equation of motion

$$m\ddot{x} - 2m\gamma(x)\dot{x} + m\omega^2 x = 0 \tag{12}$$

with $x(t=0) = x_0$ and $\dot{x}(t=0) = v_0$. The reason for only being able to solve this numerically lies in the *non-linearity* of $\gamma(x)$

$$\gamma(x) := \gamma_0 \left[1 - \frac{x(t)^2}{x_{kr}^2}\right] \tag{13}$$

appearing in the damping term (with constants $\gamma_0 > 0$ and x_{kr}). x_{kr} is a *critical displacement*, above which oscillations are damped. For displacements $x < x_{kr}$ we have $\gamma(x) < 0$, in which case the oscillations are amplified.

- Solve the initial value problem (12), (13) by employing a *Runge-Kutta algorithm* thrice with following parameters (γ_0, ω are given in units of s^{-1}, x_{kr} and x_0 in *cm* and v_0 in *cm/s*):

  1. $\omega = \pi$ $\gamma_0 = 0.025$ $x_{kr} = 0.5$
     $x_0 = 5.0$ $v_0 = 0.0$
     $0 \leq t \leq 61\,s$ TOL(rel) $= 0.00001$
  2. $\omega = \pi$ $\gamma_0 = 1.25$ $x_{kr} = 0.5$
     $x_0 = 5.0$ $v_0 = 0.0$
     $0 \leq t \leq 31\,s$ TOL(rel) $= 0.00001$
  3. $\omega = \pi$ $\gamma_0 = 2.5$ $x_{kr} = 1.0$
     $x_0 = 0.1$ $v_0 = 0.0$
     $0 \leq t \leq 31\,s$ TOL(rel) $= 0.0001$

- Plot the function $x(t)$ for all of these cases.

- Show a *phase diagram* of your results, i.e. depict the displacements $x(t)$ as functions of the velocities $\dot{x}(t)$. You will see that in all cases the oscillations heads towards a limiting curve, which is a periodic solution of the system itself. This solution, called *attractor curve*, can be approximately stated in an analytic form

$$x(t) = 2x_{kr}\cos(\omega t) - \gamma_0 \frac{x_{kr}}{2\omega}\sin(3\omega t)\ ,$$

and its respective time derivative. Plot these approximations in the same figures with your phase diagrams.[1]

- Discuss and interpret the three cases. What type of oscillations are they?

[1] More information on the van-der-Pol oscillator can be found example in: F. Scheck, *Mechanik: von den Newtonschen Gesetzen zum deterministischen Chaos*, Springer-Lehrbuch 1992, S.293f;
Ch. Gerthsen, Physik, 18. Auflage, neubearbeitet von H. Vogel, Springer, Berlin, 1995, S. 976f.