Self-consistent Neoclassical Transport Modelling for Tokamaks with E_r included

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Tokamak: self-consistent neoclassical transport coefficients

- mono-energetic drift kinetic eq. in conservative form ($p = v_\| / v$)

$$V(f) - C^p(f) = \underline{\nabla} \cdot \left(\frac{\underline{B}}{B} pv + \frac{\underline{E} \times \underline{B}}{B_0^2} \right) f + \frac{\partial}{\partial p} \dot{p} f - \frac{\nu}{2} \frac{\partial}{\partial p} (1 - p^2) \frac{\partial f}{\partial p}$$

with incompressible form of the $\underline{E} \times \underline{B}$ drift and with

$$\dot{p} = -(1 - p^2)\, v \, \frac{\underline{B} \cdot \underline{\nabla} B}{2B^2}$$

- define flux surface average, $\langle A \rangle$, and averaged moment, $[A]$:

$$\langle A \rangle = \int A \frac{d\theta}{B} \cdot \left(\int \frac{d\theta}{B} \right)^{-1} \quad \text{and} \quad [A] = \int_{-1}^{1} \langle A \rangle \, dp.$$

$$\left\langle \underline{\nabla} \cdot \frac{\underline{B}}{B} f \right\rangle = \left\langle f \frac{\underline{B} \cdot \underline{\nabla} B}{B^2} \right\rangle \quad \text{and} \quad \langle \underline{\nabla} \cdot (\underline{E} \times \underline{B}) f \rangle = \langle \underline{E} \cdot (\underline{B} \times \underline{\nabla} B) f \rangle$$

- for tokamaks: $B_\theta = \iota \varepsilon B_0$ ($\varepsilon = r/R$ and unit vector $\underline{e}_r$)

Tokamak: self-consistent neoclassical transport coefficients (2)

$$\underline{B} \cdot \underline{\nabla} B = -\iota\,\varepsilon\,(\underline{B} \times \underline{\nabla} B) \cdot \underline{e}_r$$

$\Rightarrow$ **coupling of radial flux,** $\propto (\underline{B} \times \underline{\nabla} B) \cdot \underline{e}_r$**, and of friction,** $\propto \underline{B} \cdot \underline{\nabla} B$

$$\left[\, pV(f) - pC^p(f)\,\right] = -\,\iota\varepsilon v \left[\frac{1+p^2}{2B^2}(\underline{B} \times \underline{\nabla} B) \cdot \underline{e}_r\, f\right] + \nu\left[\,pf\,\right] + \frac{1}{B_0^2}\left[(\underline{B} \times \underline{\nabla} B) \cdot \underline{E}\; pf\,\right]$$

$$\dot{r} = v_{\nabla B\,|_r} = \frac{m}{qB}\,\frac{1+p^2}{2B^2} v^2\,(\underline{B} \times \underline{\nabla} B)_r$$

- **"mixed"** $\underline{E} \times \underline{B}$**-term,** $\propto [\sin\theta\, pf]$**, not related to "simple" moments**

$\Rightarrow$ **self-consistent treatment only for** $E_r \simeq 0$

$\Rightarrow$ **flux-friction relation** $\quad (b = B/B_0)$

$$v\left[\, pV(f) - pC^p(f)\,\right] = -\iota\,\varepsilon\,\omega_{c0}\left[\, b\,\dot{r}\, f\,\right] + \nu v\left[\,pf\,\right].$$

this feature is not present in configurations with lack of symmetry

Tokamak: self-consistent neoclassical transport coefficients (3)

- **DKE for radial transport and bootstrap current** ($\hat{X} = X\,\iota v/R$)

$$\hat{V}(f_1^*) - \hat{C}^p(f_1^*) = -\rho^* \frac{1+p^2}{4\iota\varepsilon} \frac{\partial b^{-2}}{\partial\theta} \frac{d\ln f_M}{d\ln r}$$

and for ohmic current and Ware pinch

$$\hat{V}(g_1^*) - \hat{C}^p(g_1^*) = -\frac{u^*}{\iota} p$$

$f_1^* = f_1/f_M$, $g_1^* = g_1/f_M$, $\nu^* = \nu R/\iota v$ **(collisionality),**
$\rho^* = v/r\omega_{c0}$ **("gyroradius"),** $u^* = (qR/TB)\,\underline{E}\cdot\underline{B}$ **("loop voltage")**

⇒ averaged moment equations

$$-\Gamma_{11}\,(1-\alpha_{11}) + \nu^*\,\Gamma_{31} = 0$$
$$-\Gamma_{13}\,(1-\alpha_{13}) + \nu^*\,\Gamma_{33} = -\frac{2}{3}\frac{u^*}{\iota}$$

with $\Gamma_{11} = [\dot{r}^* f_1^*]$, $\Gamma_{31} = [p f_1^*]$, $\Gamma_{13} = [\dot{r}^* g_1^*]$, **and** $\Gamma_{33} = [p\, g_1^*]$
corrections $\mathcal{O}(\varepsilon)$: $\alpha_{11} = [(b-1)\,\dot{r}^* f_1^*]/[\dot{r}^* f_1^*]$, $\alpha_{13} = [(b-1)\,\dot{r}^* g_1^*]/[\dot{r}^* g_1^*]$

Tokamak: self-consistent neoclassical transport coefficients (4)

- estimate PS-contributions in Γ_{11} and Γ_{33}: ansatz $f_1^* = \phi_0 + p\phi_1$

$$\phi_1 = \frac{\rho^*}{t\varepsilon} r (f_M)' \left(1 - \frac{1}{b^2}\right) b$$

$$\Gamma_{11}{}^{PS}(1 - \alpha_{11}) = \nu^*[p^2\phi_1] = \frac{8}{3}\frac{\rho^*\nu^*}{t\varepsilon}\frac{d\ln f_M}{d\ln r}\left\langle\left(1 - \frac{1}{b^2}\right) b\right\rangle$$

- "standard model" of an elongated tokamak: $b = (1 + \kappa\varepsilon \cos\theta)^{-1}$

(reduction of the toroidal curvature, $\kappa = -b_{10}/\varepsilon$)

$$\langle(1 - b^{-2})\, b\rangle = \frac{1}{2}\kappa^2\varepsilon^2; \qquad \Gamma_{33}{}^{PS} = -\frac{1}{3t\nu^*}\langle u^*\rangle$$

- eliminate PS-contributions and "thermodynamic forces"
- use Onsager symmetry, $\hat{D}_{13} = -\hat{D}_{31}$

$\Rightarrow$ system of diffusion coefficients:

$$\left(\hat{D}_{11} - \hat{D}_{11}^{PS}\right)(1 - \alpha_{11}) - \nu^*\,\hat{D}_{31} = 0$$
$$\hat{D}_{31}\,(1 - \alpha_{13}) + \nu^*\left(\hat{D}_{33} - \hat{D}_{33}^{PS}\right) = 0$$

Tokamak: self-consistent neoclassical transport coefficients (5)

DKES database of elongated tokamak configurations:

- **3 mono-energetic transport coefficients calculated by DKES code**
- **extended "standard model":** $\underline{b} = \iota\varepsilon\underline{e}_\theta + (1 + \kappa\varepsilon\cos\theta)^{-1}\underline{e}_\phi$
- **configuration database in the ranges:**
  $0.26 \leq \iota \leq 1.04$, $0.0125 \leq \varepsilon \leq 0.4$, **and** $0.25 \leq \kappa \leq 1.0$
- **up to 35** ν/v **values** ($10^{-8} \leq \nu/v \leq 10^3$) **in 24 configurations**
- **up to 250 Fourier modes and 1000 Legendre polynomials at low** ν/v **(at very low** ν/v**, the accuracy strongly decreases)**
- **for the PS-contributions, 44 configurations are used (at** $\nu/v = 10^3$**)**
- **test functions for the non-linear fitting are constructed mainly by "trial and error" (with complete co-variance analysis)**
- **dependence on** B **and on** R **is known**
  (i.e. $B = 1$ **T and** $R = 1$ **m is used for the database)**

Tokamak: self-consistent neoclassical transport coefficients (6)

Fit results for PS-regime:

- D_{ij} in DKES notation and can be normalized by:
  $D_{11}^n = \pi/(8t)$ (analytic plateau value for $\kappa = 1$)
  $D_{31}^n = 0.9733/(t\sqrt{\varepsilon})$ (collisionless asymptote for $\kappa = 1$ and $\varepsilon \to 0$)
  $D_{33}^n = 2v/3\nu$ (collisional limit)
- for PS-contributions, best-fits are obtained by

$$D_{11}{}^{PS} = \frac{4\kappa^2}{3t^2}\frac{\nu}{v}\left(1 + 3.42\varepsilon^{3.6}(1 - 2.58t^{1.6}) - 0.6\varepsilon^2(1 - \kappa^2)\right)$$

$$D_{33}{}^{PS} = \frac{2v}{3\nu}\left(1 - 1.18(\kappa\varepsilon)^{1.84} + 0.68\varepsilon^3 t^{2.5}\right)$$

- average deviation: 0.7% for $D_{11}{}^{PS}$ and 0.1% for $D_{33}{}^{PS}$

Tokamak: self-consistent neoclassical transport coefficients (7)

Independent fit of D_{31} to the collisionless asymptote ($\nu^* \to 0$):

- **no influence of "inaccurate" test functions from other ν^*-regimes**
- **for $\kappa = 1$ and $\varepsilon \to 0$, the analytic D_{31}^n is confirmed**
- **the extension for $\kappa \neq 1$ and finite ε:**

$$D_{31}^b = 0.9733 \sqrt{\frac{\kappa}{\varepsilon \iota^2}} \left(1 - 0.67 (\kappa\varepsilon)^2\right) \cdot \left(1 + \frac{1.03}{\kappa \varepsilon^{2/3} \iota^{1/3}} \sqrt{\frac{\nu}{v}}\right)^{-1}$$

- **decreased accuracy of finite ε-correction (benchmarking with NEO: small deviation)**
- **complete disagreement with δf-MC result [Boozer, Sasinowski] (poor accuracy of δf-MC at very low ν^*)**

Tokamak: self-consistent neoclassical transport coefficients (8)

Fit results for all D_{ij} for general ν/v:

- **all 3 diffusion coefficients are fitted simultanously**
- **with the D_{31}^{b} and D_{31}^{PS}, the D_{31} is represented by**

$$D_{31} = \left({D_{31}^{b}}^{-1.75} + {D_{31}^{pl}}^{-1.75} + {D_{31}^{PS}}^{-1.75} \right)^{-1/1.75}$$

$$\text{with} \quad D_{31}^{pl} = 0.39 \frac{\kappa^2 \varepsilon}{t} \frac{v t}{\nu} \quad \text{and} \quad D_{31}^{PS} = 0.068 \frac{\kappa^2 \varepsilon}{t} \left(\frac{v t}{\nu} \right)^2$$

- **D_{11} and D_{33} from system of transport coefficients:**

$$D_{11} = {D_{11}}^{PS} + \frac{1}{\alpha} D_{31} \frac{\nu}{v}$$

$$D_{33} = {D_{33}}^{PS} - \alpha \, D_{31} \frac{v}{\nu}$$

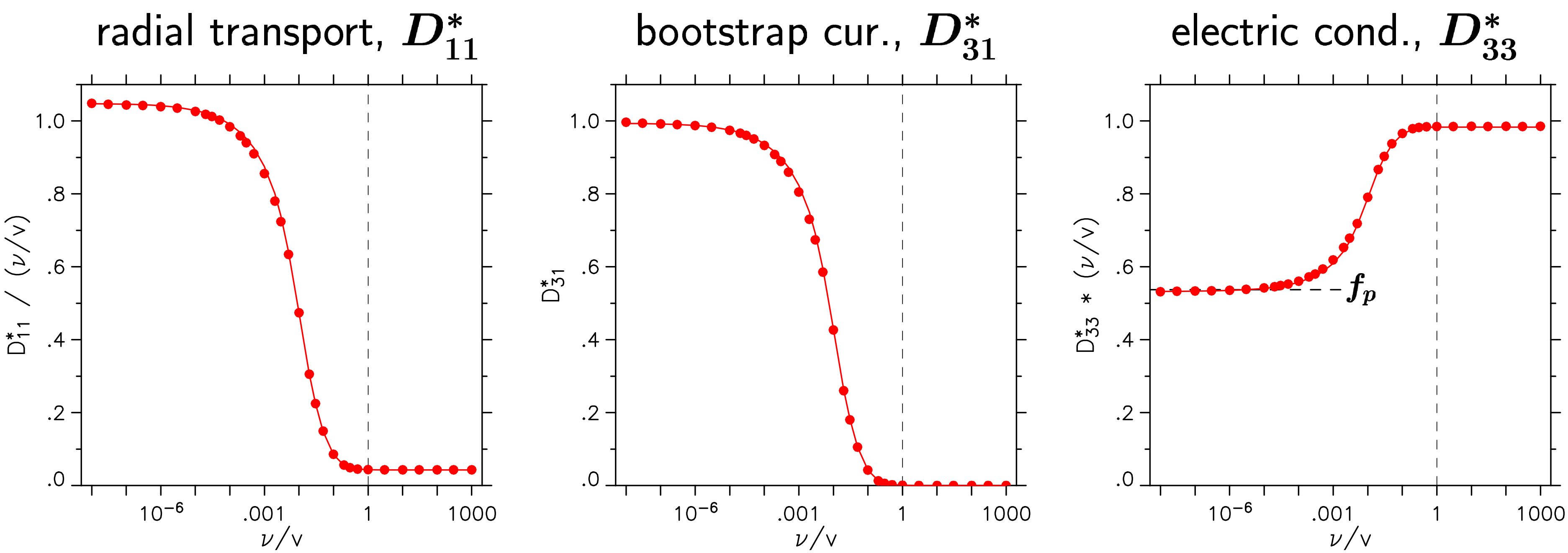
$$\text{with} \quad \alpha = t\varepsilon \left(1 - 0.97 (\kappa\varepsilon)^{1.75} \right)$$

- **average deviation of this fit to the DKES data is 1.9%**
  (deviation for D_{31} is higher, in particular for large ε)

Fitting of neoclassical transport coefficients

non-linear fitting of a database with 24 elongated tokamak configurations (DKES code)

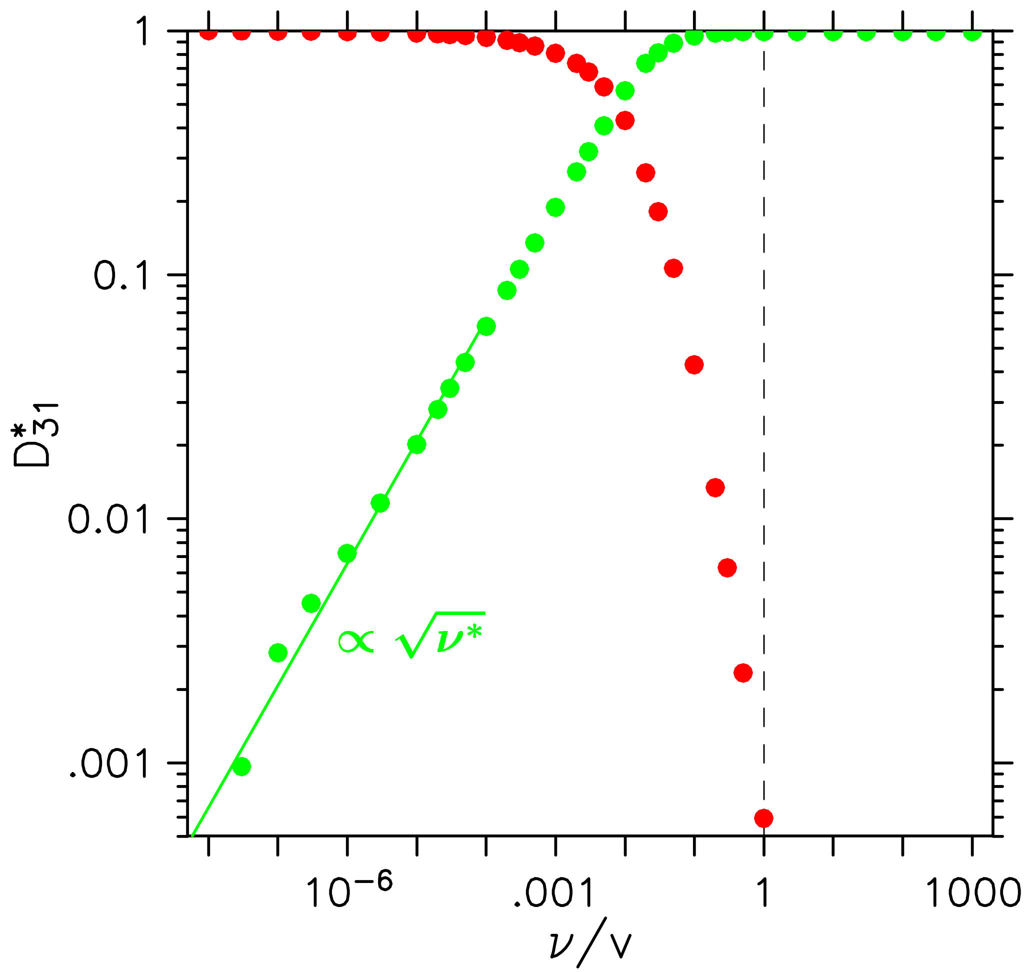
extended “standard” model: $\underline{B}/B_0 = \iota\,\varepsilon\,\underline{e}_\theta + \left(1 + \kappa\,\varepsilon\,\cos\theta\right)^{-1}\underline{e}_\varphi$



example: $\varepsilon = r/R = 0.1, \quad \iota = 0.52, \quad \kappa = 1$

Bootstrap current coefficient at very low ν^*

example: $\varepsilon = r/R = 0.1, \quad \iota = 0.52, \quad \kappa = 1$



asymptotic value for small $\varepsilon = r/R$ and $\nu^* \to 0$:

$$D^n_{31} = \frac{0.9733}{\iota}\sqrt{\frac{\kappa}{\varepsilon}}$$

$$\frac{D_{31}}{D^n_{31}} = \frac{1 + a_0}{1 + a_1\sqrt{\nu^*}}$$

$\Rightarrow \quad D_{31}/D^n_{31}$

$\Rightarrow \quad 1 - D_{31}/D^n_{31}$

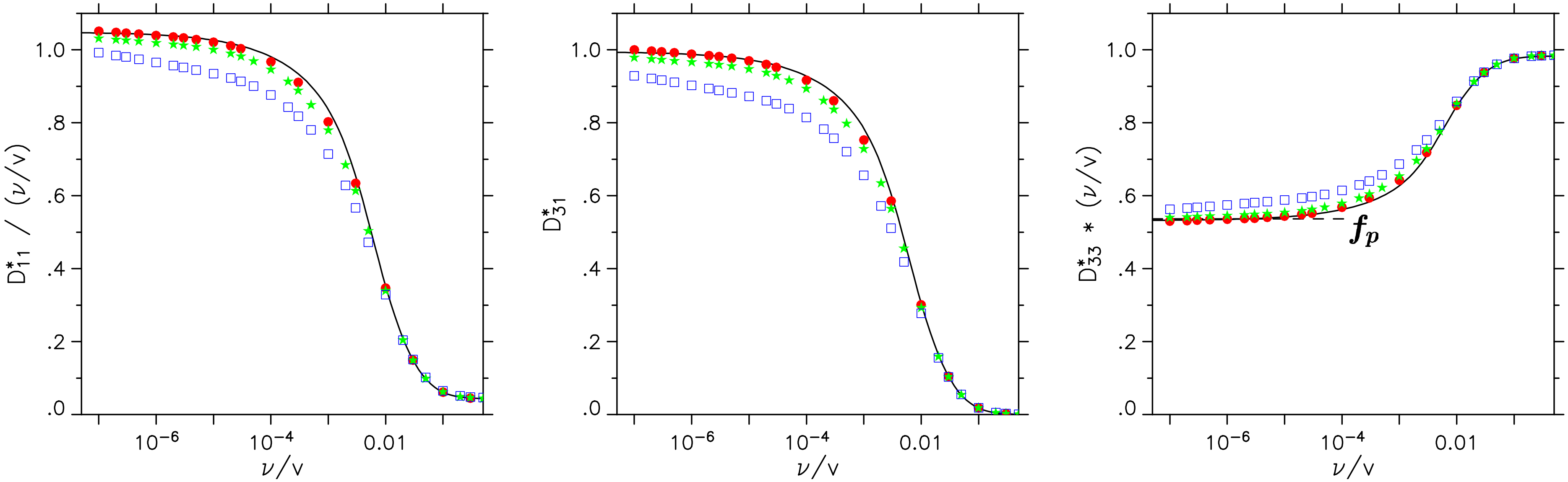
fit a_0 and a_1 in database

Neoclassical transport coefficients depending on B models

$$\frac{B}{B_0} = 1 - \varepsilon \cos\theta \qquad \frac{B}{B_0} = \frac{1}{1 + \varepsilon \cos\theta} \qquad \frac{B}{B_0} = \frac{1}{\sqrt{1 + 2\varepsilon \cos\theta + 4\varepsilon^2 \cos 2\theta}}$$

fit curve: extended "standard" model

radial transport, D^*_{11} bootstrap cur., D^*_{31} electric cond., D^*_{33}



for all B models: $\varepsilon = r/R = 0.1, \quad \iota = 0.31, \quad \kappa = 1$

PS-regime: neoclassical transport coefficients with E_r

simplified DKE: neglect mirror term $(\dot{p}\,\partial f_1/\partial p)$

$$(p+\epsilon_r^*)\frac{\partial f_1}{\partial \theta} - \frac{\nu^*}{2}\frac{\partial}{\partial p}(1-p^2)\frac{\partial}{\partial p} f_1 = \alpha^*\,(1+p^2)\,\sin\theta\; r\,\frac{d\,f_M}{dr}$$

with $\epsilon_r^* = E_r/\iota v B_0 \varepsilon$, $\nu^* = \nu R/\iota v$, $\alpha^* = \kappa v/2\iota\omega_c$, $\varepsilon = r/R$ and $\kappa = -b_{10}/\varepsilon$

ansatz: $f_1/f_M = (\varphi_0\,P_0 + \varphi_1\,P_1 + \varphi_2\,P_2 + \ldots)\cdot e^{i\theta}\cdot d\ln(f_M)/d\ln(r)$

in lowest order: $i\epsilon_r^*\,\varphi_0 + \frac{i}{3}\varphi_1 = \frac{1}{3}\alpha^*$ $\qquad (i\epsilon_r^* + \nu^*)\,\varphi_1 + \frac{i}{3}\varphi_0 = 0$

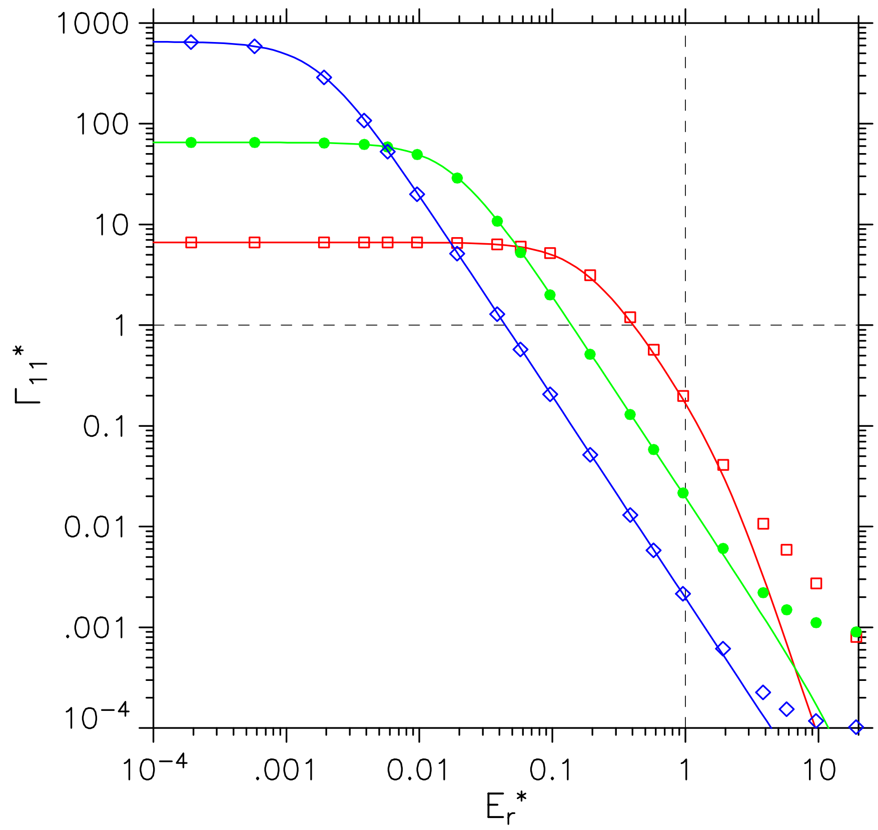
relevant for particle transport: $\Im(\varphi_0)$

$$D_{11} = \frac{4}{3}\frac{\kappa v^2}{\omega_c R}\alpha^*\,\frac{\nu^*}{(1-3\epsilon_r^{*2})^2 + (3\nu^*\epsilon_r^*)^2}$$

Tokamak: neoclassical transport coeff. in PS-regime with E_r

tokamak configuration (example): $\varepsilon = r/R = 0.1, \quad \iota = 0.52, \quad \kappa = 1$

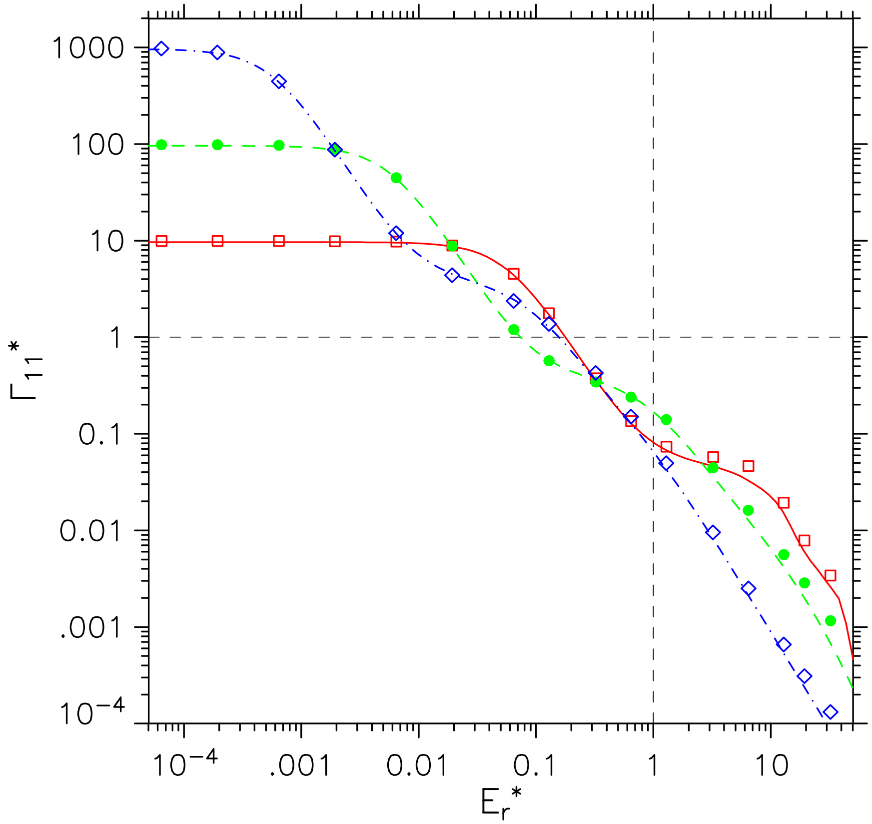
radial transport, D_{11}^*



W7-AS: neoclassical transport coeff. in PS-regime with E_r

W7-AS: $\iota \simeq 0.34$ at $r = 0.5 \cdot a$ superposition of main b_{mn} contributions

radial transport, D^*_{11}



Plateau-regime: neoclassical transport coefficients with E_r

simplified DKE: neglect mirror term ($\dot{p}\,\partial f_1/\partial p$) and use Krook's collision term

$$(p+\epsilon_r^*)\frac{\partial f_1}{\partial\theta} - \nu^* f_1 = \alpha^*\,(1+p^2)\;\sin\theta\; r\frac{d\,f_M}{dr}$$

with $\epsilon_r^* = E_r/\iota v B_0\varepsilon$, $\nu^* = \nu R/\iota v$, $\alpha^* = \kappa v/2\iota\omega_c$, $\varepsilon = r/R$ and $\kappa = -b_{10}/\varepsilon$

ansatz: $f_1/f_M = (\varphi^{\mathrm{o}} + \varphi^s \sin\theta + \varphi^c \cos\theta)\cdot d\ln(f_M)/d\ln(r)$

$$\varphi^s = \alpha^*\frac{(1+p^2)\,\nu^*}{(p+\epsilon_r^*)^2+\nu^{*2}} \qquad \text{and} \qquad \varphi^c = \alpha^*\frac{(1+p^2)(p+\epsilon_r^*)}{(p+\epsilon_r^*)^2+\nu^{*2}}$$

$\Rightarrow$ mono-energetic particle transport coefficient

$$D_{11} = \frac{\kappa v^2}{2R\omega_c}\int_{-1}^{1}(1+p^2)\,\varphi^s\;dp = \frac{\kappa v^2}{2R\omega_c}\,\alpha^*\,\nu^*\int_{-1}^{1}\frac{(1+p^2)^2}{(p+\epsilon_r^*)^2+\nu^{*2}}\;dp$$

Plateau-regime: neoclassical transport coefficients with E_r (2)

mirror term $(\dot{p}\,\partial f_1/\partial p)$ essential for bootstrap current coefficient

$$\frac{\partial}{\partial\theta}(p+\epsilon_r^*)\,\varphi^s - \frac{\kappa\varepsilon}{2}\sin^2\theta\frac{\partial}{\partial p}(1-p^2)\,\varphi^s - \frac{\nu^*}{2}\frac{\partial}{\partial p}(1-p^2)\frac{\partial}{\partial p}\varphi^{\rm o} = 0$$

with $\varphi^{\rm o}$-component independent of θ: $\langle\varphi^{\rm o}\rangle \propto p$

left term corresponds to the φ^c contribution and is negligible ($\varphi^c \propto \nu^*$ for $\nu^* \ll 1$)

$\Rightarrow$ mono-energetic bootstrap current coefficient

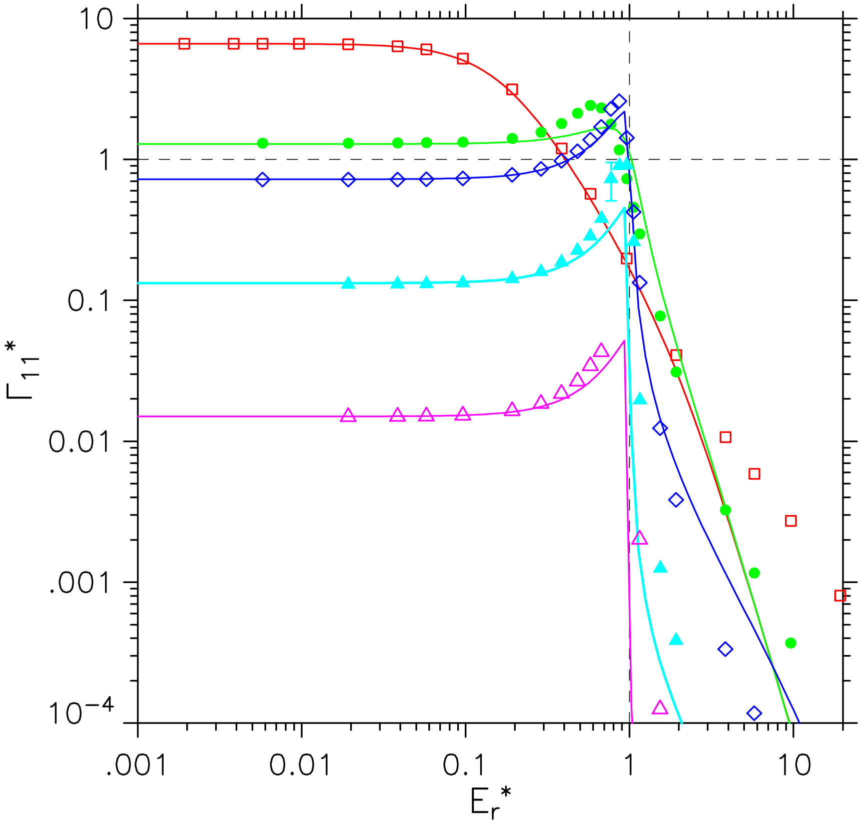
$$D_{31} = \int_{-1}^{1} p\,\langle\varphi^{\rm o}\rangle\;dp = \frac{1}{4}\,\varepsilon\,\kappa\,\alpha^* \int_{-1}^{1} \frac{(1+p^2)(1-p^2)}{(p+\epsilon_r^*)^2+\nu^{*2}}\,dp$$

in plateau-regime: $D_{31} \propto \frac{1}{\nu^*}$ for $\nu^* \ll 1$ and D_{33} independent of E_r

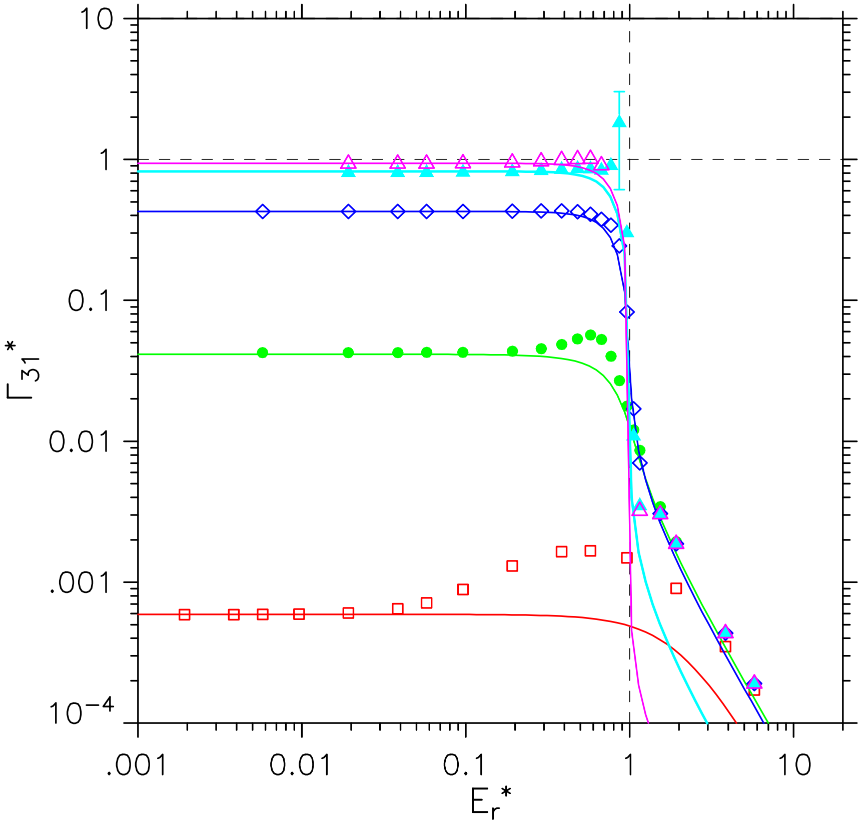
Tokamak: neoclassical transport coefficients with E_r

tokamak configuration (example): $\varepsilon = r/R = 0.1, \quad \iota = 0.52, \quad \kappa = 1$

radial transport, D_{11}^*



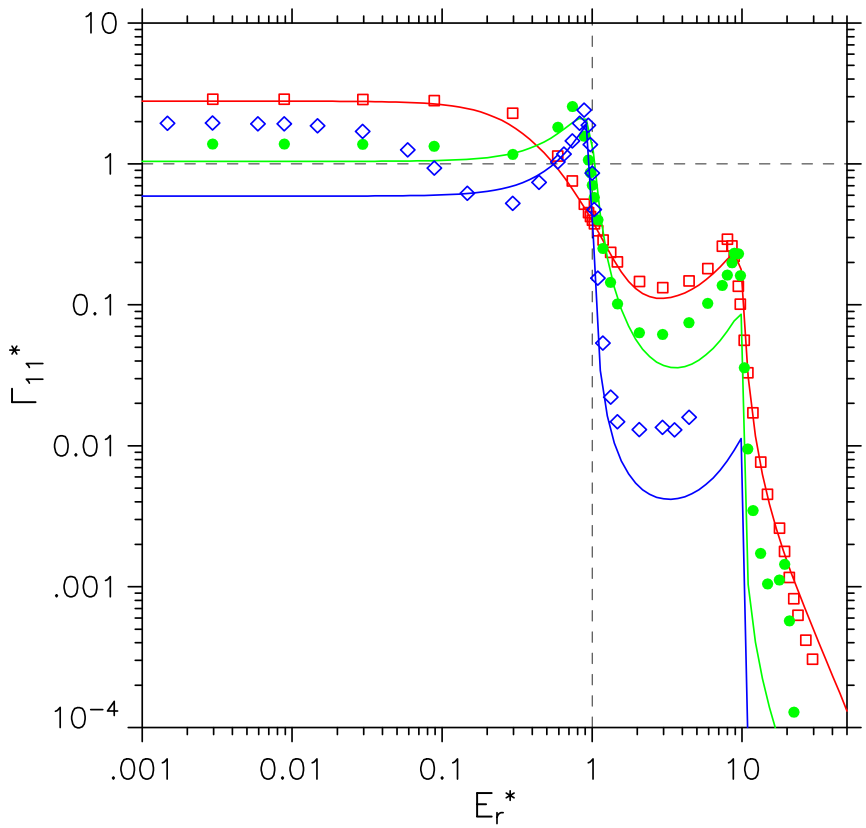
bootstrap current, D_{31}^*



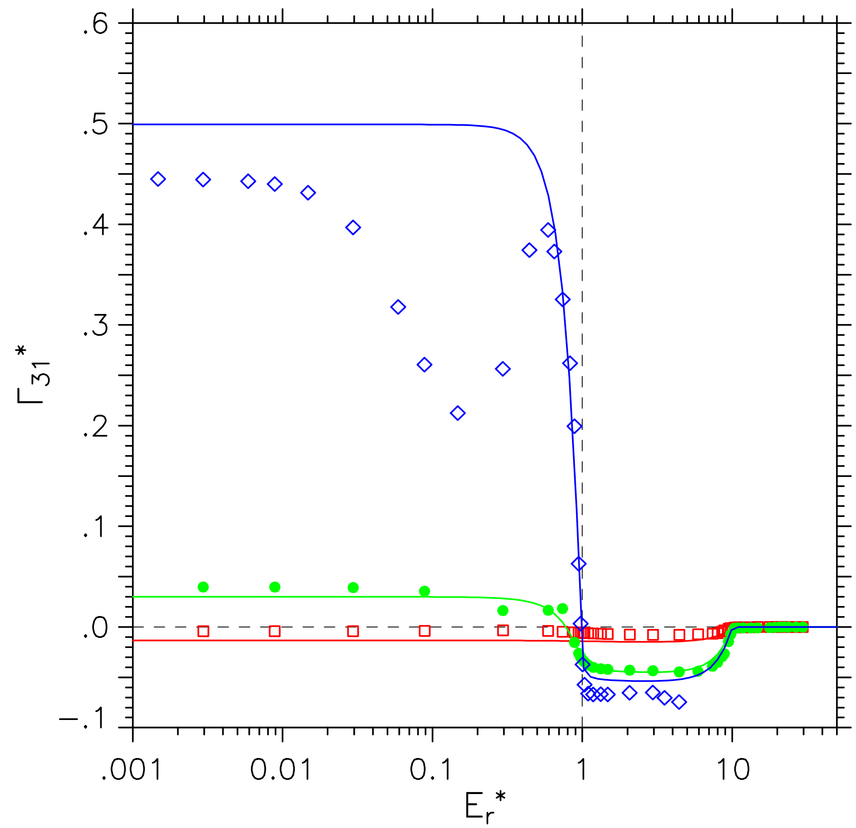
LHD-375: neoclassical transport coefficients with E_r

LHD: $R = 3.75m$ at $r = 0.5 \cdot a$ superposition of b_{10} and b_{21} contributions

radial transport, D^*_{11}



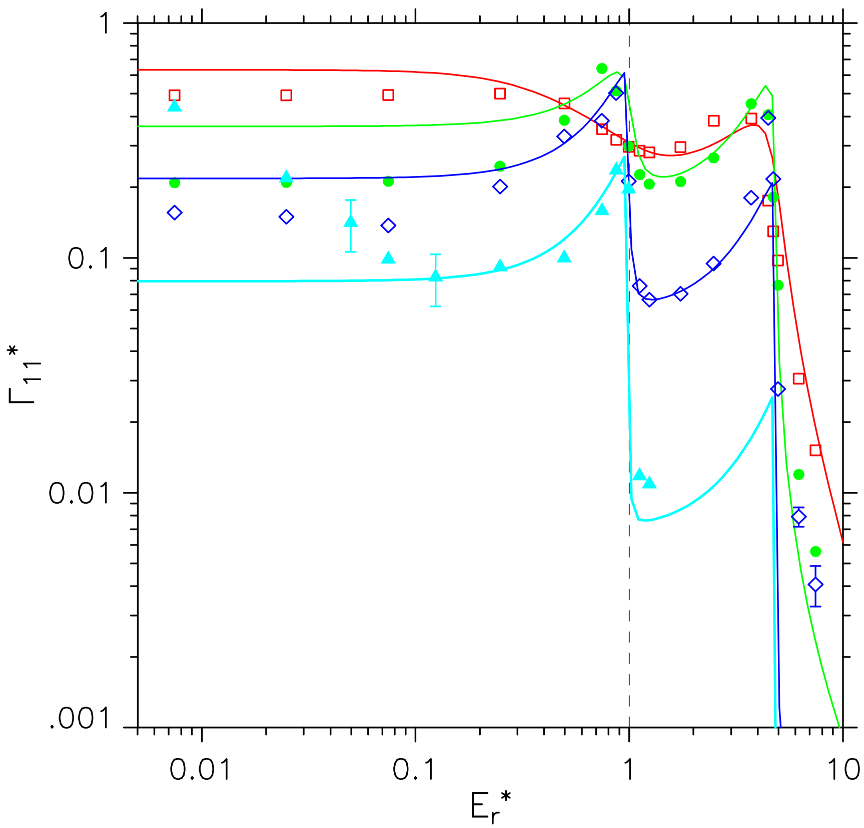
bootstrap current, D^*_{31}



W7X-sc1: neoclassical transport coefficients with E_r

W7-X "standard" at $r = 0.5 \cdot a$: superposition of b_{10} and b_{11} contributions

radial transport, D^*_{11}



bootstrap current, D^*_{31}