

Self-consistent Neoclassical Transport Modelling for Tokamaks with E_r included

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Tokamak: self-consistent neoclassical transport coefficients



ullet mono-energetic drift kinetic eq. in conservative form $(p=v_{\parallel}/v)$

$$V(f)-C^p(f)= \underline{
abla}\cdotig(rac{B}{B}pv+rac{E imes B}{B_0^2}ig)f+rac{\partial}{\partial p}\dot{p}f-rac{
u}{2}rac{\partial}{\partial p}\left(1-p^2
ight)rac{\partial f}{\partial p}$$

with incompressible form of the $\underline{E} \times \underline{B}$ drift and with

$$\dot{p} = -(1-p^2)\,v\,rac{\underline{B}\cdot \underline{
abla}B}{2B^2}$$

• define flux surface average, $\langle A \rangle$, and averaged moment, [A]:

$$\langle\,A\,
angle = \int A\,rac{d heta}{B}\cdot \Big(\intrac{d heta}{B}\Big)^{-1} \quad ext{and} \quad [\,A\,] = \int\limits_{-1}^1 \langle\,A\,
angle\,dp.$$

$$\left\langle \underline{\nabla} \cdot \frac{\underline{B}}{B} f \right\rangle = \left\langle f \frac{\underline{B} \cdot \underline{\nabla} B}{B^2} \right\rangle \text{ and } \left\langle \underline{\nabla} \cdot (\underline{E} \times \underline{B}) f \right\rangle = \left\langle \underline{E} \cdot (\underline{B} \times \underline{\nabla} B) f \right\rangle$$

• for tokamaks: $B_{\theta} = \tau \varepsilon B_0$ $(\varepsilon = r/R \text{ and unit vector } \underline{e}_r)$

Tokamak: self-consistent neoclassical transport coefficients (2)



$$\underline{B} \cdot \underline{\nabla} B = -t \varepsilon \left(\underline{B} \times \underline{\nabla} B \right) \cdot \underline{e}_r$$

 \Rightarrow coupling of radial flux, $\propto (\underline{B} \times \underline{\nabla} B) \cdot \underline{e}_r$, and of friction, $\propto \underline{B} \cdot \underline{\nabla} B$

$$egin{aligned} \left[pV(f) - pC^p(f)
ight] &= - \imath arepsilon v \left[rac{1 + p^2}{2B^2} \left(\underline{B} imes \underline{
abla} B) \cdot \underline{e}_r \, f \,
ight] +
u \left[pf \,
ight] \ &+ rac{1}{B_0^2} \left[\left(\underline{B} imes \underline{
abla} B) \cdot \underline{E} \, pf \,
ight] \ &\dot{r} = v_{
abla B|_r} = &rac{m}{aB} rac{1 + p^2}{2B^2} v^2 \, (\underline{B} imes \underline{
abla} B)_r \end{aligned}$$

- "mixed" $\underline{E} \times \underline{B}$ -term, $\propto [\sin \theta \, pf]$, not related to "simple" moments \Rightarrow self-consistent treatment only for $E_r \simeq 0$
- \Rightarrow flux-friction relation $(b = B/B_0)$

$$v\left\lceil pV(f)-pC^p(f)
ight
ceil=-arepsilon arepsilon \omega_{c0}\left\lceil b\,\dot{r}\,f
ight
ceil+
u v\left\lceil pf
ight
ceil.$$

this feature is not present in configurations with lack of symmetry

Tokamak: self-consistent neoclassical transport coefficients (3)



• DKE for radial transport and bootstrap current $(\hat{X} = X vv/R)$

$$\hat{V}(f_1^*) - \hat{C}^p(f_1^*) = -
ho^*rac{1+p^2}{4 auarepsilon}rac{\partial b^{-2}}{\partial heta}rac{d\ln f_M}{d\ln r}$$

and for ohmic current and Ware pinch

$$\hat{V}(g_1^*) - \hat{C}^p(g_1^*) = -rac{u^*}{\kappa} p^*$$

$$f_1^* = f_1/f_M, \quad g_1^* = g_1/f_M, \quad
u^* =
u R/
u \ ext{(collisionality)},
onumber $ho^* = v/r\omega_{c0} \ ext{("gyroradius")}, \quad u^* = (qR/TB)\,\underline{E}\cdot\underline{B} \ ext{("loop voltage")}$$$

 \Rightarrow averaged moment equations

$$egin{aligned} -\Gamma_{11} \left(1 - lpha_{11}
ight) +
u^* \, \Gamma_{31} &= 0 \ -\Gamma_{13} \left(1 - lpha_{13}
ight) +
u^* \, \Gamma_{33} &= -rac{2}{3} rac{u^*}{t} \end{aligned}$$

with $\Gamma_{11} = [\dot{r}^* f_1^*]$, $\Gamma_{31} = [p f_1^*]$, $\Gamma_{13} = [\dot{r}^* g_1^*]$, and $\Gamma_{33} = [p g_1^*]$ corrections $\mathcal{O}(\varepsilon)$: $\alpha_{11} = [(b-1) \, \dot{r}^* f_1^*]/[\dot{r}^* f_1^*]$, $\alpha_{13} = [(b-1) \, \dot{r}^* g_1^*]/[\dot{r}^* g_1^*]$

Tokamak: self-consistent neoclassical transport coefficients (4)



ullet estimate PS-contributions in Γ_{11} and Γ_{33} : ansatz $f_1^* = \phi_0 + p\phi_1$

$$\phi_1 = rac{
ho^*}{tarepsilon} r(f_M)' \left(1 - rac{1}{b^2}
ight) b \ \Gamma_{11}{}^{PS} (1 - lpha_{11}) =
u^* [p^2 \phi_1] = rac{8}{3} rac{
ho^*
u^*}{tarepsilon} rac{d \ln f_M}{d \ln r} \left\langle \left(1 - rac{1}{b^2}
ight) b
ight
angle$$

• "standard model" of an elongated tokamak: $b = (1 + \kappa \varepsilon \cos \theta)^{-1}$ (reduction of the toroidal curvature, $\kappa = -b_{10}/\varepsilon$)

$$\langle \left(1-b^{-2}
ight)b
angle = rac{1}{2}\,\kappa^2arepsilon^2; \qquad \qquad \Gamma_{33}{}^{PS} = -rac{1}{3\imath
u^*}\langle u^*
angle$$

- eliminate PS-contributions and "thermodynamic forces"
- use Onsager symmetry, $\hat{D}_{13} = -\hat{D}_{31}$
- \Rightarrow system of diffusion coefficients:

$$egin{split} \left(\hat{D}_{11} - \hat{D}_{11}^{PS}
ight) \left(1 - lpha_{11}
ight) -
u^* \,\hat{D}_{31} &= 0 \ \hat{D}_{31} \left(1 - lpha_{13}
ight) +
u^* \left(\hat{D}_{33} - \hat{D}_{33}^{PS}
ight) &= 0 \end{split}$$

Tokamak: self-consistent neoclassical transport coefficients (5)



DKES database of elongated tokamak configurations:

- 3 mono-energetic transport coefficients calculated by DKES code
- extended "standard model": $\underline{b} = \varepsilon \underline{e}_{\theta} + (1 + \kappa \varepsilon \cos \theta)^{-1} \underline{e}_{\phi}$
- configuration database in the ranges: $0.26 \le t \le 1.04, \, 0.0125 \le \varepsilon \le 0.4, \, \text{and} \, 0.25 \le \kappa \le 1.0$
- up to 35 ν/v values $(10^{-8} \le \nu/v \le 10^3)$ in 24 configurations
- up to 250 Fourier modes and 1000 Legendre polynomials at low ν/v (at very low ν/v , the accuracy strongly decreases)
- for the PS-contributions, 44 configurations are used (at $\nu/v = 10^3$)
- test functions for the non-linear fitting are constructed mainly by "trial and error" (with complete co-variance analysis)
- dependence on B and on R is known (i.e. B = 1 T and R = 1 m is used for the database)

Tokamak: self-consistent neoclassical transport coefficients (6)



Fit results for PS-regime:

• D_{ij} in DKES notation and can be normalized by:

$$D_{11}^n=\pi/(8t)$$
 (analytic plateau value for $\kappa=1$) $D_{31}^n=0.9733/(t\sqrt{\varepsilon})$ (collisionless asymptote for $\kappa=1$ and $\varepsilon\to 0$) $D_{33}^n=2v/3\nu$ (collisional limit)

• for PS-contributions, best-fits are obtained by

$$D_{11}{}^{PS} = rac{4\kappa^2}{3 au^2}rac{
u}{v}\left(1 + 3.42arepsilon^{3.6}(1 - 2.58arepsilon^{1.6}) - 0.6arepsilon^2(1 - \kappa^2)
ight) \ D_{33}{}^{PS} = rac{2v}{3
u}\left(1 - 1.18(\kappaarepsilon)^{1.84} + 0.68arepsilon^3arepsilon^{2.5}
ight)$$

ullet average deviation: 0.7% for ${D_{11}}^{PS}$ and 0.1% for ${D_{33}}^{PS}$

Tokamak: self-consistent neoclassical transport coefficients (7)



Independent fit of D_{31} to the collisionless asymptote $(\nu^* \to 0)$:

- no influence of "inaccurate" test functions from other ν^* -regimes
- for $\kappa = 1$ and $\varepsilon \to 0$, the analytic D_{31}^n is confirmed
- the extension for $\kappa \neq 1$ and finite ε :

$$D_{31}^b = 0.9733 \sqrt{rac{\kappa}{arepsilon t^2}} \left(1 - 0.67 (\kappa arepsilon)^2
ight) \cdot \left(1 + rac{1.03}{\kappa arepsilon^{2/3} t^{1/3}} \sqrt{rac{
u}{v}}
ight)^{-1}$$

- decreased accuracy of finite ε -correction (benchmarking with NEO: small deviation)
- complete disagreement with δf -MC result [Boozer, Sasinowski] (poor accuracy of δf -MC at very low ν^*)

Tokamak: self-consistent neoclassical transport coefficients (8)



Fit results for all D_{ij} for general ν/v :

- all 3 diffusion coefficients are fitted simultanously
- with the D_{31}^b and D_{31}^{PS} , the D_{31} is represented by

$$D_{31} = \left(D_{31}^{b^{-1.75}} + D_{31}^{pl^{-1.75}} + D_{31}^{PS^{-1.75}} \right)^{-1/1.75}$$
 with $D_{31}^{pl} = 0.39 rac{\kappa^2 \varepsilon}{\tau} rac{v au}{
u}$ and $D_{31}^{PS} = 0.068 rac{\kappa^2 \varepsilon}{ au} \left(rac{v au}{
u}
ight)^2$

• D_{11} and D_{33} from system of transport coefficients:

$$D_{11} = D_{11}^{PS} + rac{1}{lpha}D_{31}rac{
u}{v} \ D_{33} = D_{33}^{PS} - lpha\,D_{31}rac{
u}{v} \ ext{with} \quad lpha = tarepsilonig(1-0.97(\kappaarepsilon)^{1.75}ig)$$

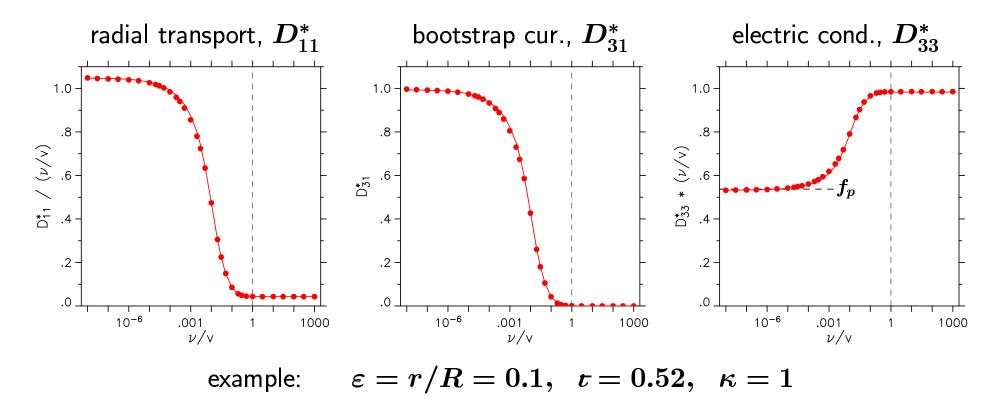
• average deviation of this fit to the DKES data is 1.9% (deviation for D_{31} is higher, in particular for large ε)

Fitting of neoclassical transport coefficients



non-linear fitting of a database with 24 elongated tokamak configurations (DKES code)

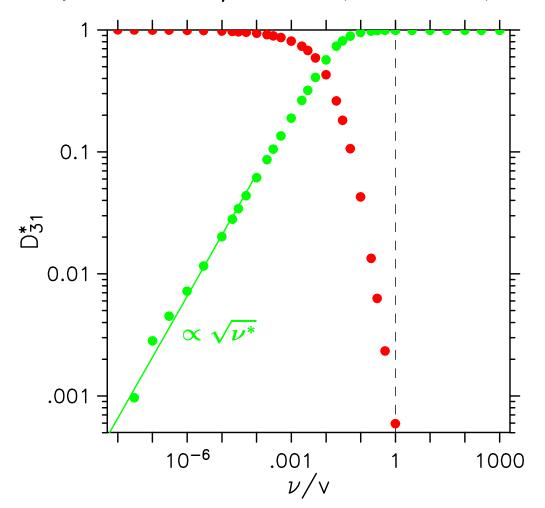
extended "standard" model:
$$\underline{B}/B_0 = arepsilon arepsilon \, \underline{e}_ heta + ig(1 + \kappa \, arepsilon \, \cos hetaig)^{-1} \underline{e}_arphi$$



Bootstrap current coefficient at very low u^*



example: $\varepsilon=r/R=0.1,\ \epsilon=0.52,\ \kappa=1$



asymptotic value for small arepsilon=r/R and $u^* o 0$:

$$egin{align} D_{31}^{\,n} &= rac{0.9733}{arepsilon} \sqrt{rac{\kappa}{arepsilon}} \ rac{D_{31}}{D_{31}^{\,n}} &= rac{1 + a_0}{1 + a_1 \sqrt{
u^*}} \ \end{array}$$

$$\Rightarrow D_{31}/D_{31}^n$$

$$\Rightarrow 1 - D_{31}/D_{31}^n$$

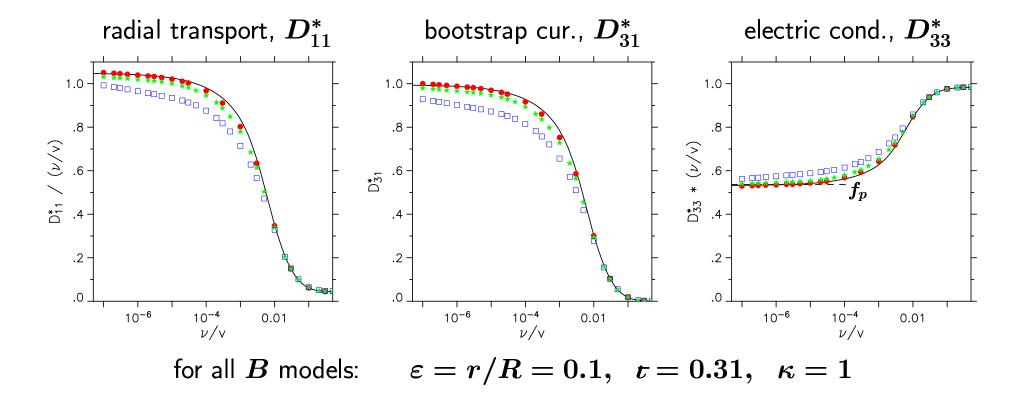
fit a_0 and a_1 in database

Neoclassical transport coefficients depending on $oldsymbol{B}$ models



$$rac{B}{B_0} = 1 - arepsilon \cos heta \qquad rac{B}{B_0} = rac{1}{1 + arepsilon \cos heta} \qquad rac{B}{B_0} = rac{1}{\sqrt{1 + 2arepsilon \cos heta + 4arepsilon^2 \cos 2 heta}}$$

fit curve: extended "standard" model



PS-regime: neoclassical transport coefficients with $oldsymbol{E_r}$



simplified DKE: neglect mirror term $(\dot{p}\,\partial f_1/\partial p)$

$$(p+\epsilon_r^*)rac{\partial f_1}{\partial heta} - rac{
u^*}{2}rac{\partial}{\partial p}\left(1-p^2
ight)rac{\partial}{\partial p}\,f_1 = lpha^*\left(1+p^2
ight)\,\sin heta\,\,r\,rac{d\,f_M}{dr}$$

with $\epsilon_r^*=E_r/\epsilon v B_0 arepsilon$, $u^*=
u R/\epsilon v$, $\alpha^*=\kappa v/2\epsilon \omega_c$, arepsilon=r/R and $\kappa=-b_{10}/arepsilon$ ansatz: $f_1/f_M=(arphi_0\,P_0+arphi_1\,P_1+arphi_2\,P_2+...)\cdot e^{i heta}\cdot d\ln(f_M)/d\ln(r)$

in lowest order: $i\epsilon_r^*\,arphi_0+rac{i}{3}\,arphi_1=rac{1}{3}\,lpha^*$ $(i\epsilon_r^*+
u^*)\,arphi_1+rac{i}{3}\,arphi_0=0$

relevant for particle transport: $\Im(arphi_0)$

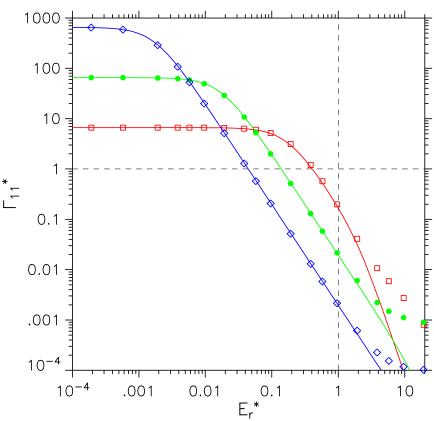
$$D_{11} = rac{4}{3} rac{\kappa v^2}{\omega_c R} \, lpha^* \, rac{
u^*}{(1 - 3 \epsilon_r^{*2})^2 + (3
u^* \epsilon_r^*)^2}$$

Tokamak: neoclassical transport coeff. in PS-regime with $oldsymbol{E_r}$



tokamak configuration (example): $\varepsilon = r/R = 0.1, \;\; t = 0.52, \;\; \kappa = 1$

radial transport, D_{11}^st



W7-AS: neoclassical transport coeff. in PS-regime with $oldsymbol{E_r}$



W7-AS: $r \simeq 0.34$ at $r = 0.5 \cdot a$ superposition of main b_{mn} contributions

radial transport, D_{11}^st 1000 100 10 0.1 0.01 .001 10^{-4} .001 10

Plateau-regime: neoclassical transport coefficients with $oldsymbol{E_r}$



simplified DKE: neglect mirror term $(\dot{p}\,\partial f_1/\partial p)$ and use Krook's collision term

$$(p+\epsilon_r^*)rac{\partial f_1}{\partial heta} -
u^*f_1 = lpha^*\left(1+p^2
ight)\,\sin heta\,\, rrac{d\,f_M}{dr}$$

with $\epsilon_r^*=E_r/vvB_0arepsilon$, $u^*=
uR/vv$, $\alpha^*=\kappa v/2v\omega_c$, arepsilon=r/R and $\kappa=-b_{10}/arepsilon$ ansatz: $f_1/f_M=(arphi^{
m o}+arphi^s\sin heta+arphi^c\cos heta)\cdot d\ln(f_M)/d\ln(r)$

$$arphi^s = lpha^* rac{(1+p^2)\,
u^*}{(p+\epsilon_r^*)^2 +
u^{*2}} \qquad ext{and} \qquad arphi^c = lpha^* rac{(1+p^2)(p+\epsilon_r^*)}{(p+\epsilon_r^*)^2 +
u^{*2}}$$

⇒ mono-energetic particle transport coefficient

$$D_{11} = rac{\kappa v^2}{2R\omega_c} \int\limits_{-1}^{1} (1+p^2) \, arphi^s \; dp = rac{\kappa v^2}{2R\omega_c} lpha^* \,
u^* \int\limits_{-1}^{1} rac{(1+p^2)^2}{(p+\epsilon_r^*)^2 +
u^{*2}} \, dp$$

Plateau-regime: neoclassical transport coefficients with $m{E_r}$ (2)



mirror term $(\dot{p}\,\partial f_1/\partial p)$ essential for bootstrap current coefficient

$$rac{\partial}{\partial heta} \left(p + \epsilon_r^*
ight) arphi^s - rac{\kappa arepsilon}{2} \sin^2 heta rac{\partial}{\partial p} \left(1 - p^2
ight) arphi^s - rac{
u^*}{2} rac{\partial}{\partial p} \left(1 - p^2
ight) rac{\partial}{\partial p} arphi^{
m o} = 0$$

with $\varphi^{\rm o}$ -component independent of θ : $\langle \varphi^{\rm o} \rangle \propto p$ left term corresponds to the φ^c contribution and is negligible $(\varphi^c \propto \nu^* \text{ for } \nu^* \ll 1)$

⇒ mono-energetic bootstrap current coefficient

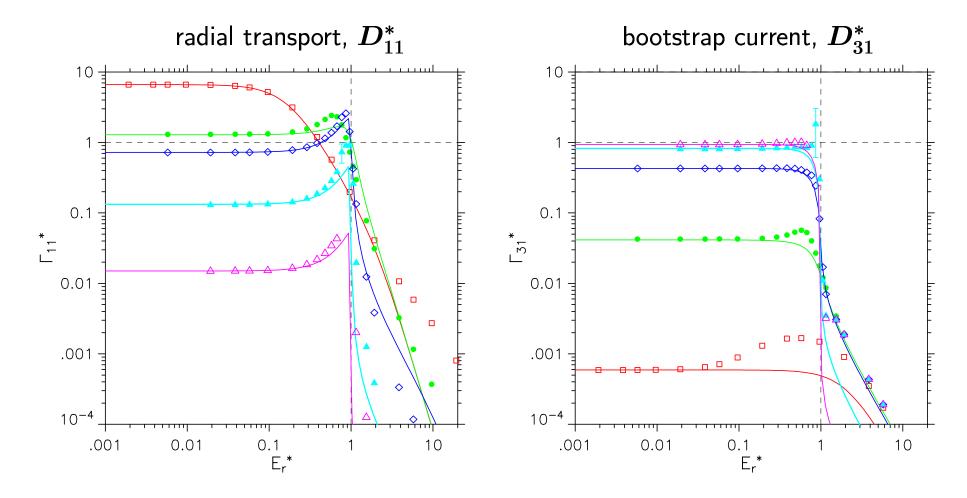
$$D_{31} = \int\limits_{-1}^{1} p raket{arphi^{
m o}} dp = rac{1}{4} arepsilon \, \kappa \, lpha^* \int\limits_{-1}^{1} rac{(1+p^2)(1-p^2)}{(p+\epsilon_r^*)^2 +
u^{*2}} \, dp$$

in plateau-regime: $D_{31} \propto rac{1}{
u^*}$ for $u^* \ll 1$ and D_{33} independent of E_r

Tokamak: neoclassical transport coefficients with $oldsymbol{E_r}$



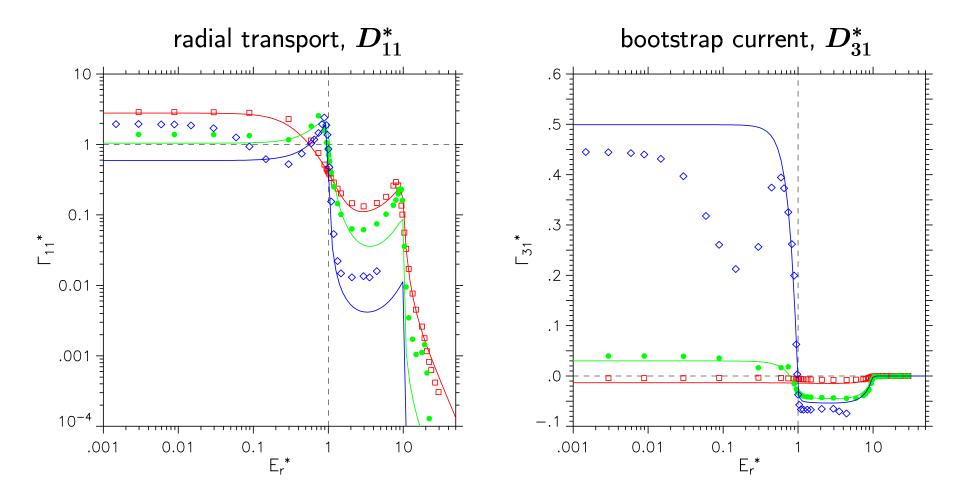
tokamak configuration (example):
$$\varepsilon=r/R=0.1, \;\; t=0.52, \;\; \kappa=1$$



LHD-375: neoclassical transport coefficients with $\boldsymbol{E_r}$



LHD: R=3.75m at $r=0.5 \cdot a$ superposition of b_{10} and b_{21} contributions



W7X-sc1: neoclassical transport coefficients with $oldsymbol{E_r}$



W7-X "standard" at $r=0.5 \cdot a$: superposition of b_{10} and b_{11} contributions

