The Shortcomings of the Conventional Ripple Average for Use in Determining "Parallel" Transport Coefficients and Their Possible Amelioration

C.D. Beidler

Max-Planck-Institut für Plasmaphysik
IPP–EURATOM Association
Greifswald, GERMANY

Introduction

Although capable of providing accurate and efficient results for the radial transport coefficient in many stellarators, use of the ripple average to describe the bootstrap current has not proven successful ....

Fundamental loss of relevant information due to ripple average?

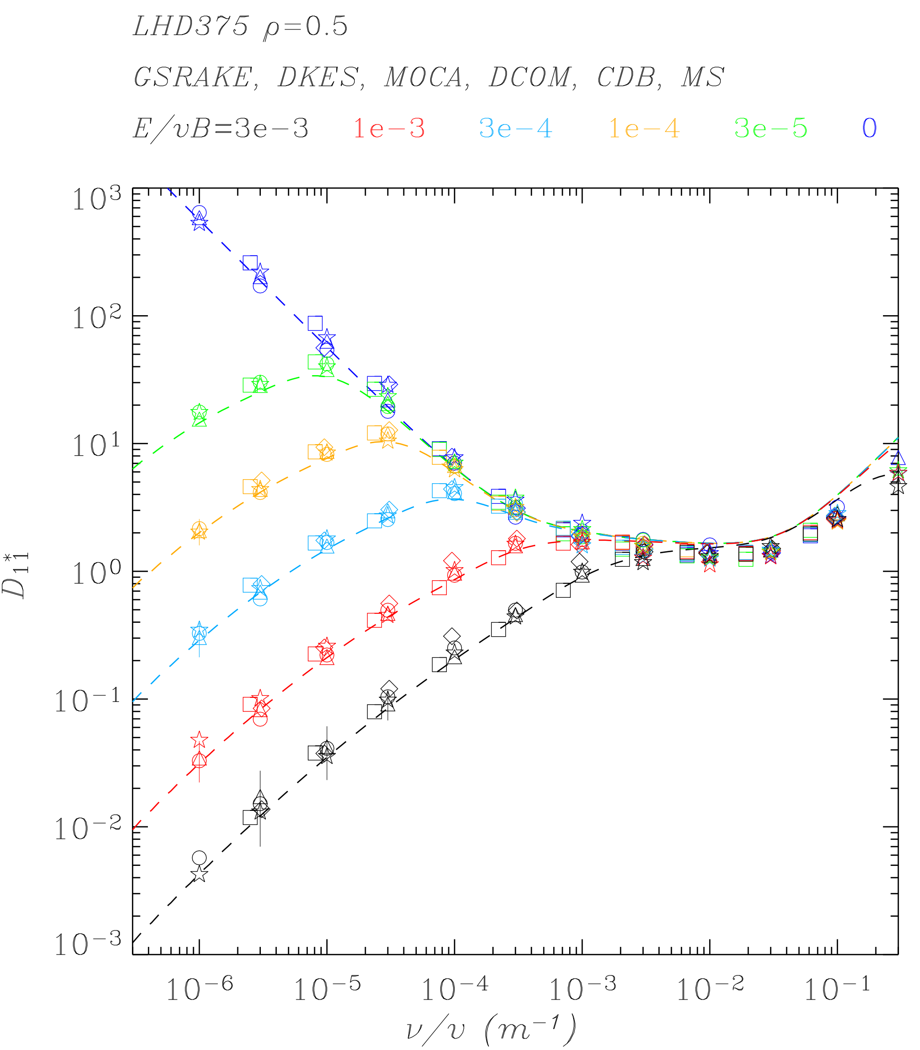
$\rightarrow$ unlikely due to success of NEO?

Simplifying assumptions inappropriate for non-localized particles?

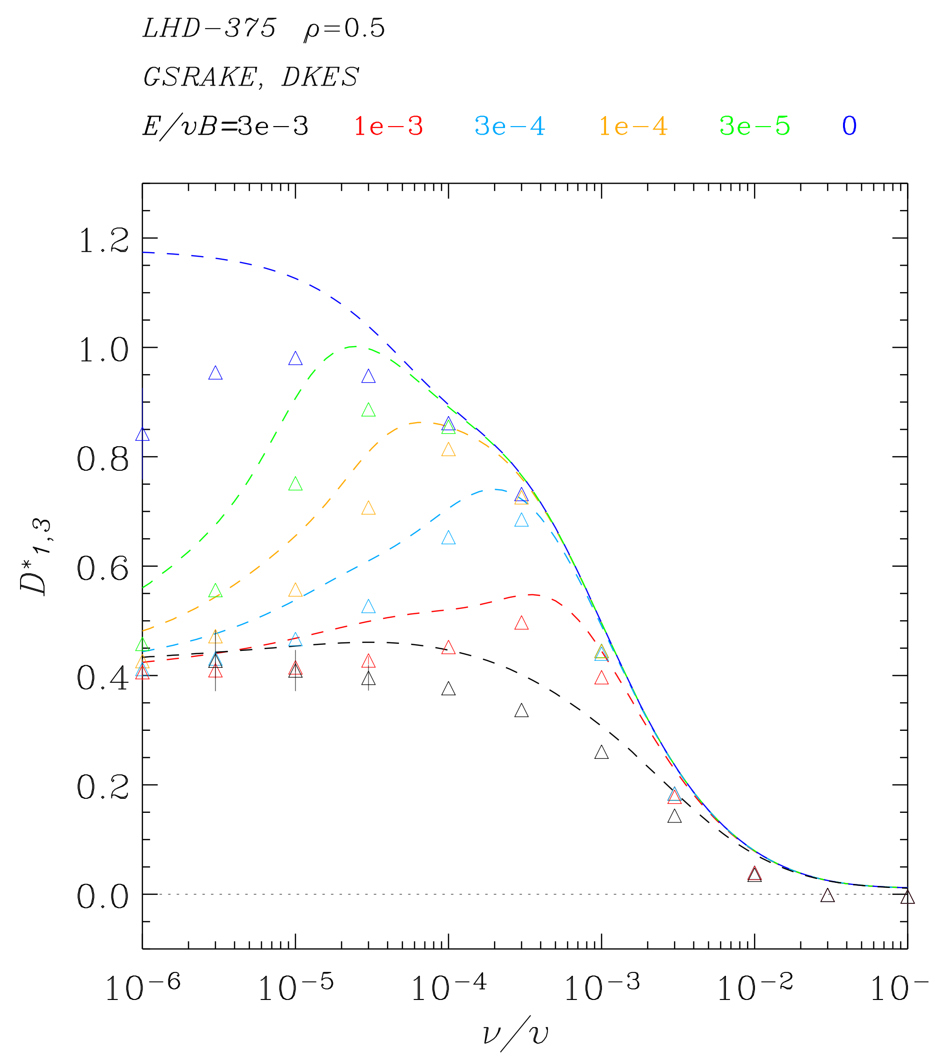
$\rightarrow$ average along ζ instead of truly along field line

$\rightarrow$ error small in ι/N for localized particles

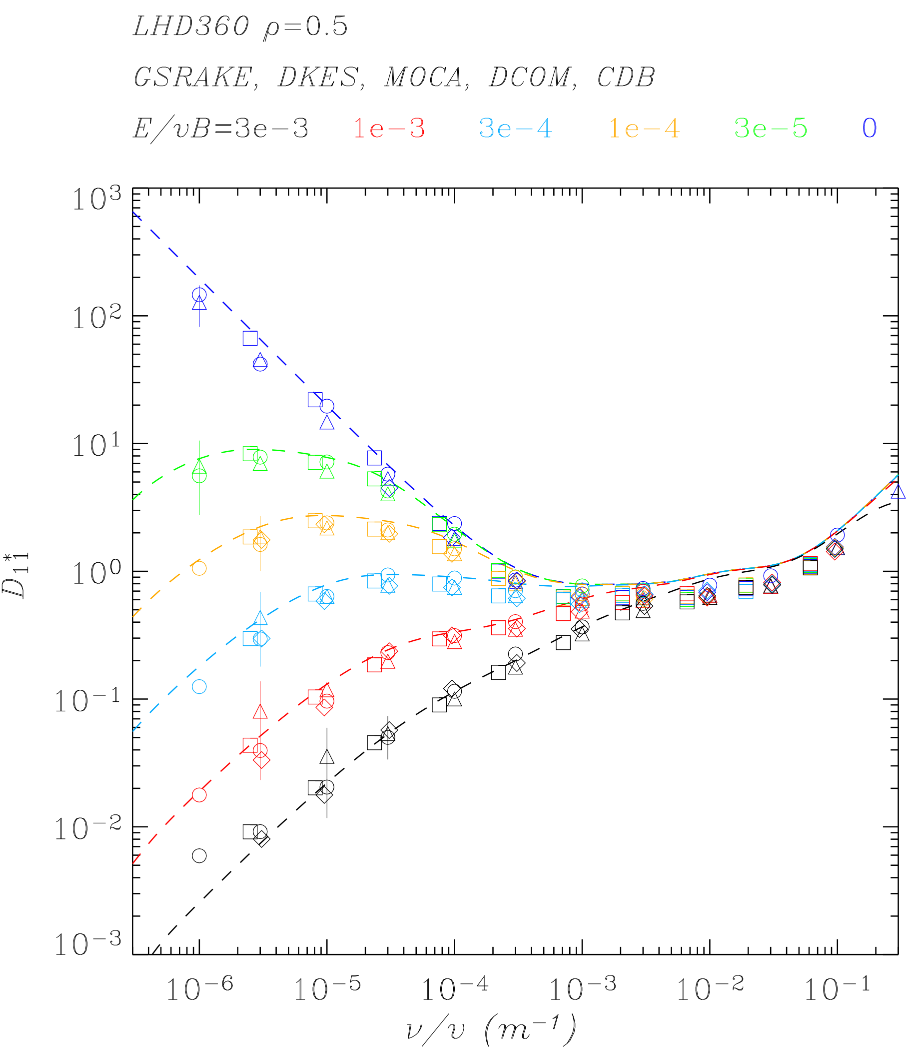
$\rightarrow$ similar for non-localized particles?

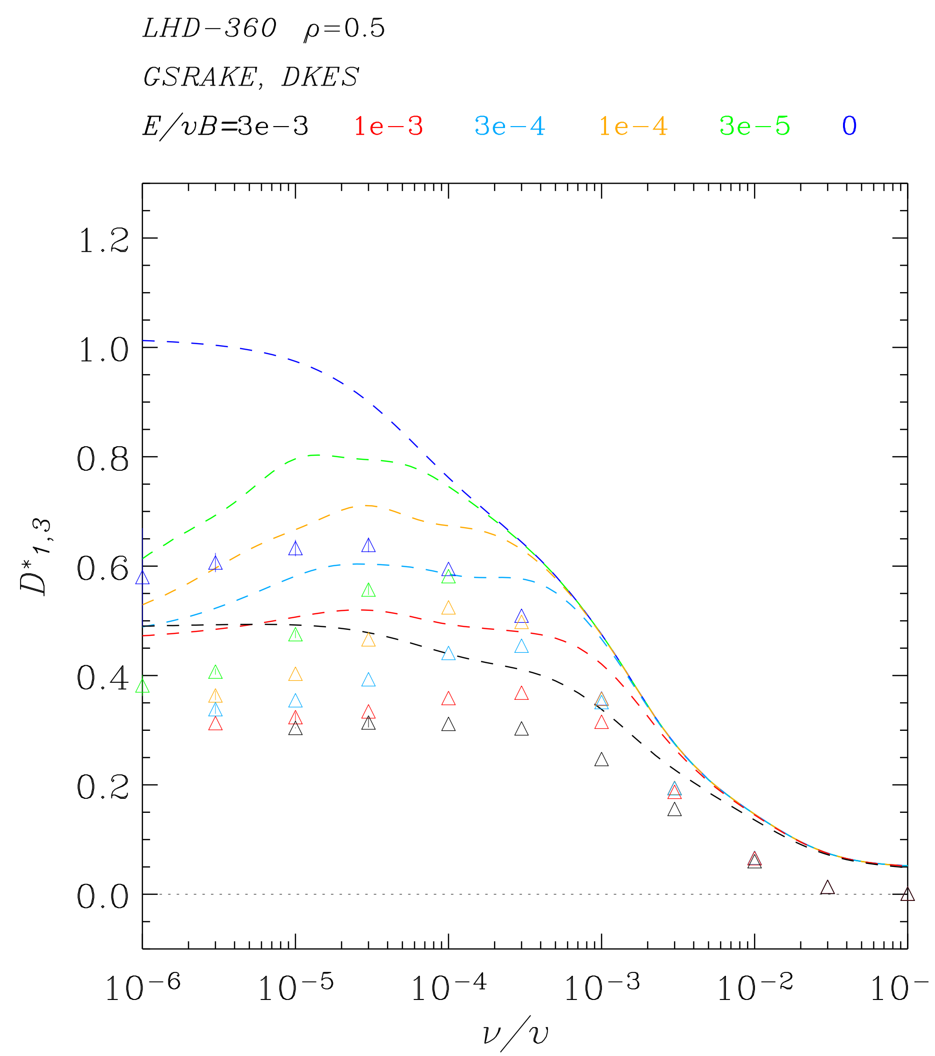


LHD375 ρ=0.5
GSRAKE, DKES, MOCA, DCOM, CDB, MS
E/vB=3e−3 1e−3 3e−4 1e−4 3e−5 0
D_{11}^*
ν/v (m⁻¹)



LHD−375 ρ=0.5
GSRAKE, DKES
E/vB=3e−3 1e−3 3e−4 1e−4 3e−5 0
$D^*_{1,3}$
ν/v


LHD360 ρ=0.5
GSRAKE, DKES, MOCA, DCOM, CDB
E/vB=3e−3 1e−3 3e−4 1e−4 3e−5 0
D*11
ν/v (m−1)


LHD−360 ρ=0.5
GSRAKE, DKES
E/vB=3e−3 1e−3 3e−4 1e−4 3e−5 0
D*1,3
ν/v

Basic Assumptions

"Local" ansatz including resultant variation of density on a flux surface

$$n = n_0(r) + n_1(r,\theta,\zeta) \qquad \longrightarrow \qquad \Phi = \Phi_0(r) + \Phi_1(r,\theta,\zeta)$$

favors linearization of the kinetic equation using $f = f_0 + f_1$, with

$$f_0 = n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp(-\kappa_0/kT)$$

where

$$\kappa_0 = \kappa_0(r) \equiv \mathcal{E} - q\Phi_0 = \kappa + q\Phi_1$$

The terminology, "mono-energetic", implies a single value of κ_0.

The Linearized Drift Kinetic Equation – 1

Linearized DKE $(r, \zeta = N\phi, \theta, \kappa_0, \mu)$ in Boozer toroidal coordinates

$$\frac{\mathrm{d}r}{\mathrm{d}t}\left\{\frac{n_0'}{n_0} + \frac{q\Phi_0'}{kT} + \left(\frac{\kappa_0}{kT} - \frac{3}{2}\right)\frac{T'}{T}\right\} f_0 + \frac{q}{kT}\frac{\partial V_\zeta}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t} f_0 + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial f_1}{\partial \zeta} + \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\partial f_1}{\partial \theta} = C_\mu(f_1)$$

letting

$$f_1 = R_0\left\{\frac{n_0'}{n_0} + \frac{q\Phi_0'}{kT} + \left(\frac{\kappa_0}{kT} - \frac{3}{2}\right)\frac{T'}{T}\right\} f_0\,\hat{f} + \frac{qV_L}{kT} f_0\,\hat{g}$$

$$\frac{1}{R_0}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial \hat{f}}{\partial \zeta} + \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\partial \hat{f}}{\partial \theta} = C_\mu(\hat{f})$$

$$\frac{1}{V_L}\frac{\partial V_\zeta}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t} + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial \hat{g}}{\partial \zeta} + \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\partial \hat{g}}{\partial \theta} = C_\mu(\hat{g})$$

The Linearized Drift Kinetic Equation – 2

with drift equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{rB_0}\frac{\partial \Phi}{\partial \theta} + \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial \theta} \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = N\frac{\lambda v B}{g}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\iota}{N}\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \frac{1}{rB_0}\frac{\partial \Phi}{\partial r} - \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial r}$$

$$v_d = \frac{\kappa}{qR_0B_0} \qquad \epsilon_t = \frac{r}{R_0} \qquad \lambda = \frac{v_{||}}{v} \qquad g \approx R_0B_0$$

Separation of time scales in these variables requires the assumption of "small" rotational transform per field period (typically, $\iota/N = 0$).

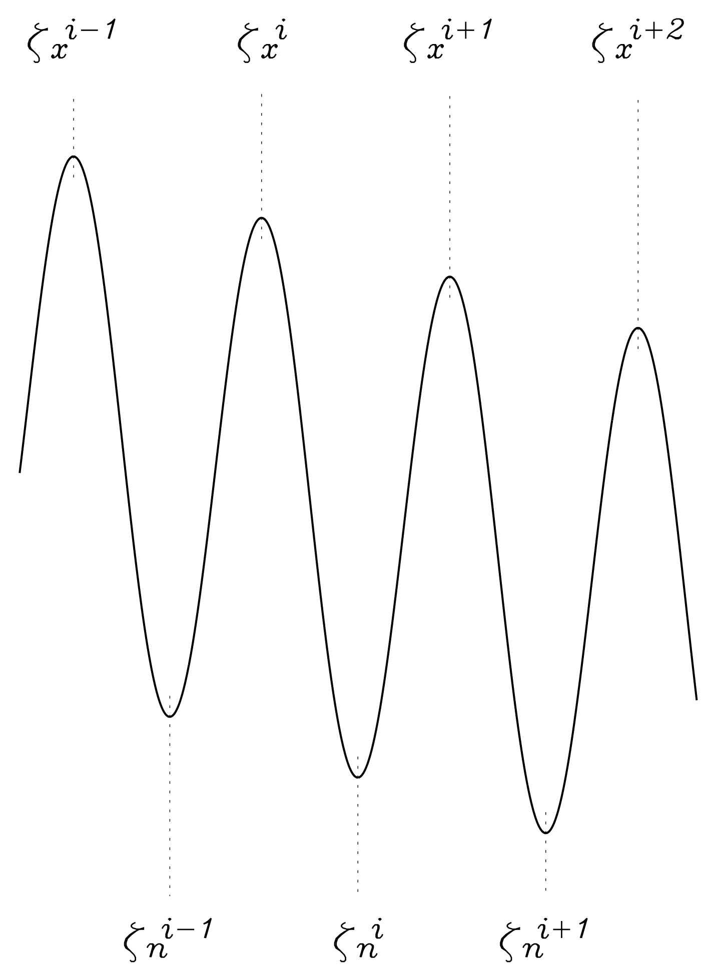
Change of Variables – 1

The assumption $\iota/N = 0$ may be avoided by introducing a change of variables. In Boozer coordinates field lines are straight

$$\theta = \theta_0 + \iota\zeta/N$$

Supposing a local minimum in B to be found at $\theta = \theta_n^i$, $\zeta = \zeta_n^i$, a simple change of reference point yields

$$\theta = \theta_n^i + (\iota/N)(\zeta - \zeta_n^i)$$



Change of Variables – 2

Local-field-line variable, θ_n, defined so that

$$\theta_n \equiv \theta_n^i + (\iota/N)\, H_n(\zeta) \qquad\qquad \zeta_n \equiv \zeta_n^i + H_n(\zeta)$$

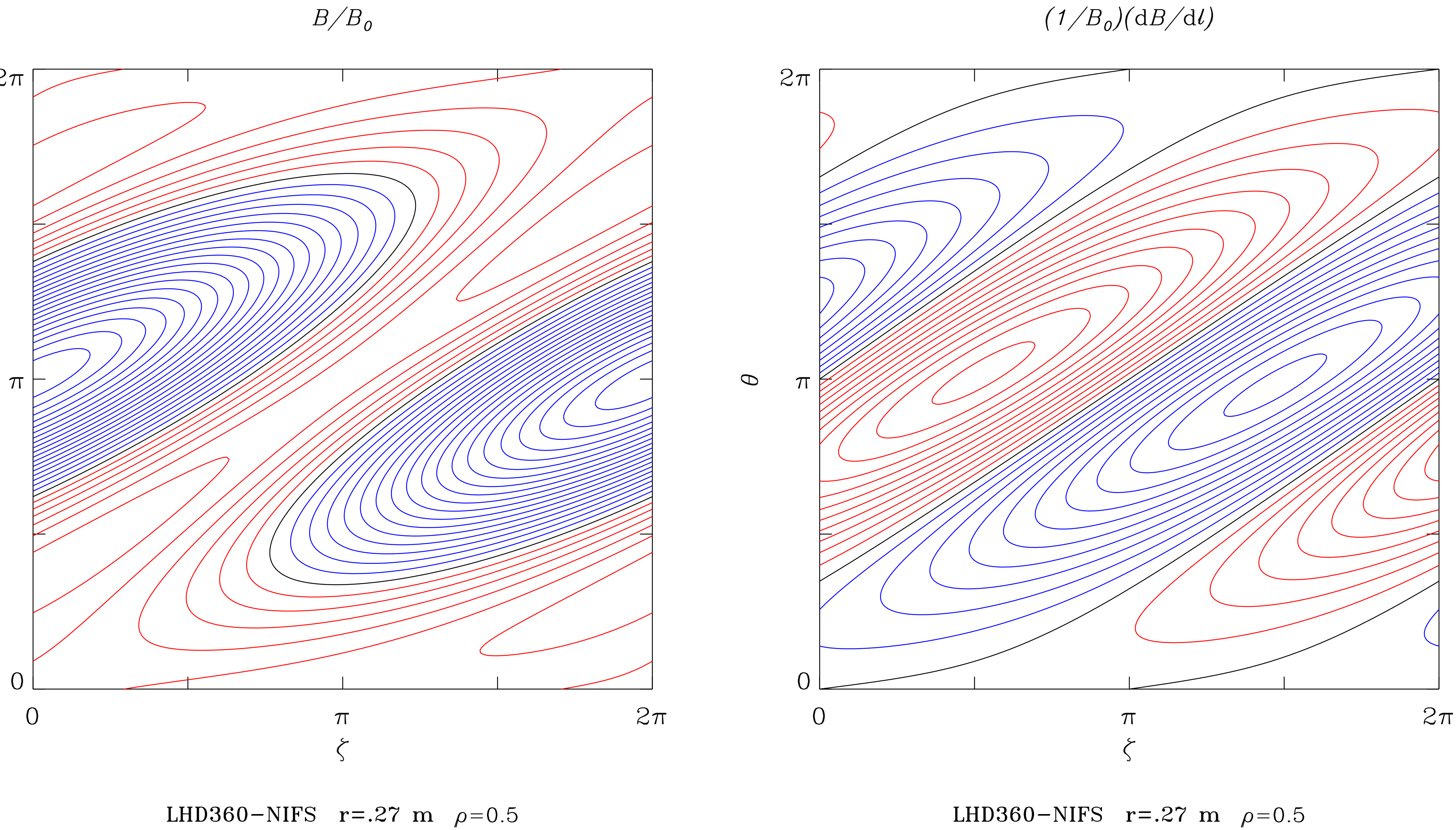
$$\Longrightarrow \qquad \theta = \theta_n + (\iota/N)(\zeta - \zeta_n)$$

where H_n is the infinite sum of Heaviside functions (minimum selector)

$$H_n(y) = \sum_{j=1}^{\infty} (y_n^{i+j} - y_n^{i+j-1})\, \mathbb{H}(\zeta - \zeta_x^{i+j}) + (y_n^{i-j} - y_n^{i-j+1})\, \mathbb{H}(\zeta_x^{i-j+1} - \zeta)$$

Advantages

- Within a single ripple, θ_n is constant along a field line.
- For “multiple-helicity” B, each value of θ_n in the range $0 \le \theta_n \le 2\pi$ defines a unique ripple.



B/B_0
$(1/B_0)(\mathrm{d}B/\mathrm{d}\ell)$
θ
0
π
2π
ζ
LHD360−NIFS r=.27 m ρ=0.5
LHD360−NIFS r=.27 m ρ=0.5

Local Model Field

B can be expressed in terms of θ_n using

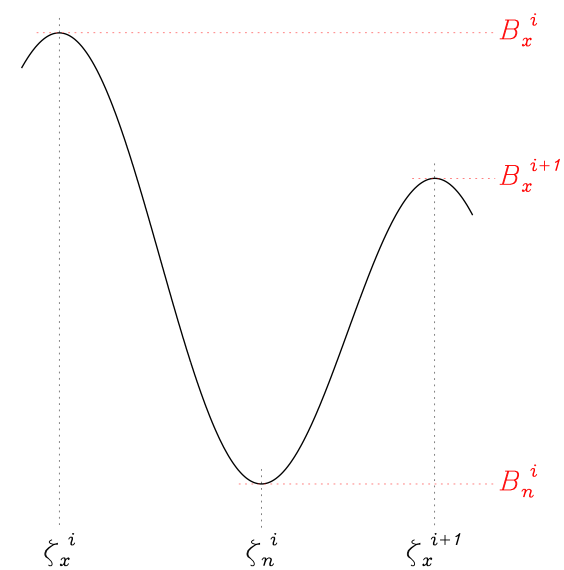
$$\frac{B}{B_0} = \frac{B_x}{2B_0}(1-\cos\eta) + \frac{B_n}{2B_0}(1+\cos\eta) \qquad \eta = \frac{(\zeta-\zeta_n)\pi}{|\zeta_n-\zeta_x|}$$

with

$$B_n \equiv B_n^i + H_n(B)$$

$$\zeta_x \equiv \zeta_x^i + H_x(\zeta)$$

$$B_x \equiv B_x^i + H_x(B)$$



and where (maximum selector)

$$H_x(y) = \sum_{j=1}^{\infty}(y_x^{i+j} - y_x^{i+j-1})\, I\!H(\zeta - \zeta_n^{i+j-1}) + (y_x^{i-j} - y_x^{i-j+1})\, I\!H(\zeta_n^{i-j} - \zeta)$$

Local Variables

Local pitch-angle variable

$$k^2 = \frac{\kappa/\mu - B_n}{2B_0\delta} \qquad\qquad 2B_0\delta = \min(B_x) - B_n$$

Coordinate change $(\zeta,\, \theta,\, \mu) \to (\zeta,\, \theta_n,\, k^2)$ leads to

$$\frac{1}{R_0}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial \hat{f}}{\partial \zeta} + \frac{\mathrm{d}\theta_n}{\mathrm{d}t}\frac{\partial \hat{f}}{\partial \theta_n} + \frac{\mathrm{d}k^2}{\mathrm{d}t}\frac{\partial \hat{f}}{\partial k^2} = C_{k^2}(\hat{f})$$

$$\frac{1}{V_L}\frac{\partial V_\zeta}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t} + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial \hat{g}}{\partial \zeta} + \frac{\mathrm{d}\theta_n}{\mathrm{d}t}\frac{\partial \hat{g}}{\partial \theta_n} + \frac{\mathrm{d}k^2}{\mathrm{d}t}\frac{\partial \hat{g}}{\partial k^2} = C_{k^2}(\hat{g})$$

The Drift Equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial\theta_n}{\partial\theta}\left\{\frac{1}{rB_0}\frac{\partial\Phi}{\partial\theta_n} + \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial\theta_n}\right\} \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = N\frac{\lambda v B}{g}$$

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \frac{\partial\theta_n}{\partial\theta}\left\{\frac{\iota\!\!\!-}{N}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial\zeta_n}{\partial\zeta} - \frac{1}{rB_0}\frac{\partial\Phi}{\partial r} - \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial r}\right\}$$

$$\begin{aligned}\frac{\mathrm{d}k^2}{\mathrm{d}t} = &-\frac{1}{2\delta}\left\{\frac{q}{\mu B_0}\frac{\partial\Phi}{\partial r} + \frac{1}{B_0}\frac{\partial B_n}{\partial r} + 2k^2\frac{\partial\delta}{\partial r}\right\}\frac{\mathrm{d}r}{\mathrm{d}t}\\ &-\frac{1}{2\delta}\left\{\frac{q}{\mu B_0}\frac{\partial\Phi}{\partial\zeta} + \frac{1}{B_0}\frac{\partial B_n}{\partial\zeta} + 2k^2\frac{\partial\delta}{\partial\zeta}\right\}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\\ &-\frac{1}{2\delta}\left\{\frac{q}{\mu B_0}\frac{\partial\Phi}{\partial\theta_n} + \frac{1}{B_0}\frac{\partial B_n}{\partial\theta_n} + 2k^2\frac{\partial\delta}{\partial\theta_n}\right\}\frac{\mathrm{d}\theta_n}{\mathrm{d}t}\end{aligned}$$

The Ripple Average – 1

Separation of time scales may now be carried out using the operator

$$\langle x \rangle = \frac{1}{\tau} \int_{-\eta_1^*}^{\eta_2^*} \mathrm{d}\zeta \left| \frac{\mathrm{d}\zeta}{\mathrm{d}t} \right|^{-1} x \qquad \tau = \int_{-\eta_1^*}^{\eta_2^*} \mathrm{d}\zeta \left| \frac{\mathrm{d}\zeta}{\mathrm{d}t} \right|^{-1}$$

with the limits on the integration given by

$$\eta_j^* = \begin{cases} 2 \sin^{-1} k_j & k_j^2 \leq 1 \\ \pi & k_j^2 \geq 1 \end{cases}$$

Note that the integration is carried out over only "half" of the bounce time for localized particles.

The Ripple Average – 2

The ripple-averaged kinetic equations

$$\frac{1}{R_0}\left\langle\frac{\mathrm{d}r}{\mathrm{d}t}\right\rangle + \left\langle\frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial\hat{f}}{\partial\zeta}\right\rangle + \left\langle\frac{\mathrm{d}\theta_n}{\mathrm{d}t}\right\rangle\frac{\partial\hat{f}}{\partial\theta_n} + \left\langle\frac{\mathrm{d}k^2}{\mathrm{d}t}\right\rangle\frac{\partial\hat{f}}{\partial k^2} = \left\langle C_{k^2}(\hat{f})\right\rangle$$

$$\frac{1}{V_L}\left\langle\frac{\partial V_\zeta}{\partial\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right\rangle + \left\langle\frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial\hat{g}}{\partial\zeta}\right\rangle + \left\langle\frac{\mathrm{d}\theta_n}{\mathrm{d}t}\right\rangle\frac{\partial\hat{g}}{\partial\theta_n} + \left\langle\frac{\mathrm{d}k^2}{\mathrm{d}t}\right\rangle\frac{\partial\hat{g}}{\partial k^2} = \left\langle C_{k^2}(\hat{g})\right\rangle$$

are obtained assuming the local variation of $\hat{f}$ and $\hat{g}$ may be ignored during the averaging process, e.g.

$$\hat{f} = \hat{f}^i + H_n(\hat{f})$$

This is also done for Φ_1 and V_ζ.

The Ripple Average – 3

Taking

$$\mathbb{H}(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$$

one obtains

$$\left\langle \frac{\mathrm{d}\zeta}{\mathrm{d}t} \frac{\partial \hat{f}}{\partial \zeta} \right\rangle = \frac{\Delta \hat{f}}{\tau}$$

where

$$\Delta x = \begin{cases} 0 & k^2 < 1 \\ (x^{i+1} - x^{i-1})/2 & k^2 \geq 1 \end{cases}$$

The conventional ripple average ignores this term — for $\iota/N = 0$ one obtains $\theta_n^{i+1} = \theta_n^{i-1}$ and thus $\Delta \hat{f} = 0$.

The Ripple Average – 4

Compare with the "fast" part of

$$\left\langle \frac{\mathrm{d}\theta_n}{\mathrm{d}t} \right\rangle \frac{\partial \hat{f}}{\partial \theta_n} = \frac{\partial \theta_n}{\partial \theta} \frac{\iota}{N} \frac{\Delta \zeta_n}{\tau} \frac{\partial \hat{f}}{\partial \theta_n}$$

which does **not** vanish in the conventional approach since $\Delta\zeta_n = 2\pi$ and $\tau \propto 1/N$.

Using $\Delta\theta_n = (\iota/N)\Delta\zeta_n$, one has

$$\left\langle \frac{\mathrm{d}\theta_n}{\mathrm{d}t} \right\rangle \frac{\partial \hat{f}}{\partial \theta_n} = \frac{\partial \theta_n}{\partial \theta} \frac{\Delta \theta_n}{\tau} \frac{\partial \hat{f}}{\partial \theta_n}$$

which is of the same order as

$$\left\langle \frac{\mathrm{d}\zeta}{\mathrm{d}t} \frac{\partial \hat{f}}{\partial \zeta} \right\rangle = \frac{\Delta \hat{f}}{\tau}$$

The Ripple Average – 5

$$\left\langle \frac{\mathrm{d}r}{\mathrm{d}t} \right\rangle = \frac{\partial \theta_n}{\partial \theta} \left\{ \frac{1}{rB_0} \frac{\partial \Phi_1}{\partial \theta_n} + \frac{v_d}{\epsilon_t} \left\langle (1+\lambda^2) \frac{1}{B} \frac{\partial B}{\partial \theta_n} \right\rangle \right\}$$

$$\left\langle \frac{\mathrm{d}\theta_n}{\mathrm{d}t} \right\rangle = \frac{\partial \theta_n}{\partial \theta} \left\{ \frac{\Delta \theta_n}{\tau} - \frac{1}{rB_0} \frac{\partial \Phi}{\partial r} - \frac{v_d}{\epsilon_t} \left\langle (1+\lambda^2) \frac{1}{B} \frac{\partial B}{\partial r} \right\rangle \right\}$$

$$\left\langle \frac{\mathrm{d}k^2}{\mathrm{d}t} \right\rangle = -\frac{1}{2\delta} \left\{ \frac{q}{\mu B_0} \frac{\partial \Phi}{\partial r} + \frac{1}{B_0} \frac{\partial B_n}{\partial r} + 2k^2 \frac{\partial \delta}{\partial r} \right\} \left\langle \frac{\mathrm{d}r}{\mathrm{d}t} \right\rangle$$
$$-\frac{1}{2\delta} \left\{ \frac{q}{\mu B_0} \frac{\Delta \Phi_1}{\tau} + \frac{1}{B_0} \frac{\Delta B_n}{\tau} + 2k^2 \frac{\Delta \delta}{\tau} \right\}$$
$$-\frac{1}{2\delta} \left\{ \frac{q}{\mu B_0} \frac{\partial \Phi_1}{\partial \theta_n} + \frac{1}{B_0} \frac{\partial B_n}{\partial \theta_n} + 2k^2 \frac{\partial \delta}{\partial \theta_n} \right\} \left\langle \frac{\mathrm{d}\theta_n}{\mathrm{d}t} \right\rangle$$

Summary

The common assumption $\iota/N \to 0$, used to simplify the ripple average, leads to errors of

- $\mathcal{O}(\iota/N)$ for localized particles
- $\mathcal{O}(1)$ for non-localized particles

The approach presented here

- truly performs the ripple average along a field line
- employs "local" variables akin to those of the conventional theory
- treats localized and non-localized particles identically (effects of current diffusion on the localized population?)
- is capable of handling variation of Φ on flux surfaces

But ... how to best handle particles which are reflected but not localized?