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Optimal inhomogeneity for superconductivity

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We study the effect of nonuniform transverse couplings on a quasi-one-dimensional superconductor. We show that inhomogeneous couplings quite generally increase the superconducting (pairing) gap relative to the uniform system, but that beyond an ‘‘optimal’’ degree of inhomogeneity, they lead to a suppression of the tendency towards phase coherence. The optimal conditions for superconductivity are derived. We also show that a *delocalized*, spin-gapped phase is stable against weak disorder in a four-leg ladder with moderate repulsive interactions.

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A number of experiments[1] including neutron scattering, angle-resolved photoemission, and scanning tunneling microscopy[2] suggest that many high-$T_c$superconductors have a highly inhomogeneous electronic structure which often has a quasi-one-dimensional nature—‘‘stripes.’’ It is still unclear whether inhomogeneities, and stripes in particular, are an essential feature of high-$T_c$superconductivity.

There is compelling *theoretical* evidence that quasi-one-dimensional systems, such as two-leg Hubbard or t-J ladders (2LL's), have a strong tendency towards the formation of a spin gap and substantial superconducting (SC) pair-field correlations.[3] However, the gap size and the tendency to superconductivity tend to decrease rapidly with increasing width for ladders with more legs.[4,5] Due to the phenomenological similarity between 2LL's and two-dimensional cuprates, it has been suggested that the pairing mechanism for superconductivity in the cuprates may have a quasi-one-dimensional origin.[6] However, strong pairing is often accompanied by a small superfluid stiffness, especially in quasi-one-dimensional systems, so a large spin gap does not necessarily imply a high T_c.[6–8] In other words, while local pairing develops easily in a 2LL, it is hard for the *phase* of the pairs on different weakly coupled 2LL's to develop coherence. Another problem with quasi-one-dimensional superconductors is their extreme sensitivity to disorder.[9] Since weak disorder does not destroy superconductivity in two dimensions, it is intuitively clear that this sensitivity is mitigated if one increases the number of coupled chains. Taken together, these observations suggest that there exists an intermediate ‘‘optimal’’ degree of inhomogeneity[5,10] which maximizes T_c.

In this paper, we analyze the relation between inhomogeneity and superconductivity in ‘‘microscopic’’ inhomogeneous multileg ladder Hubbard models in which the SC gap arises solely as a result of *repulsive* interactions between electrons. This allows us to make *quantitative* estimates (at least for weak coupling) of the optimal degree of inhomogeneity without *ad hoc* assumptions. In addition, we show that for a four-leg ladder (4LL), there is a broad range of repulsive interactions for which disorder is irrelevant in the renormalization-group sense.

Specifically, we have carried out a weak-coupling renormalization-group (RG) analysis of a model of two 2LL's (see point *a* below) with a repulsive on-site interaction U coupled via an interladder hopping t and with an on-site energy offset ε between the 2LL's. We consider t in the range $1 \geqslant t > 0$; in the units we have adopted $t=1$ and $\varepsilon=0$ corresponds to a homogeneous 4LL. We have explored the behavior of this model as a function of ε and t; for simplicity, we will here report only on the results along a one parameter cut through parameter space, $\varepsilon=\varepsilon_0(1-t^2)$, which interpolates between the homogeneous 4LL at $t=1$, and two, inequivalent, decoupled 2LL's at $t=0$. Moreover, we have considered both the case of open (OBC) and of periodic (PBC) boundary conditions in the direction transverse to the chains.

Our main results are presented in Fig. 1, and can be summarized as follows: In the parameter range that we have considered, the 4LL is in a ‘‘superconducting’’ phase characterized by very slowly decaying SC correlations. There are two gap scales associated with this phase, a ‘‘pseudogap’’ Δ_{PG} associated with pairing of holes within one pair of bands, and a second scale, which we call the superconducting gap Δ_{SC}, associated with phase coherence between Cooper pairs on different pairs of bands. The crucial result (see Fig. 1) is that Δ_{SC} is not a monotonic function of t, and, in particular, it has an absolute maximum at a certain $t=t_{opt} < 1$, i.e., for an *inhomogeneous* 4LL. Therefore, superconductivity can be optimized by introducing a certain degree of inhomogeneity.

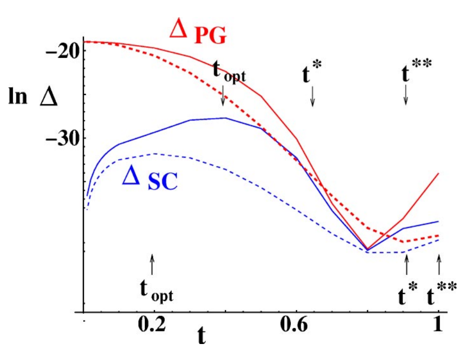

FIG. 1. (Color online) Logarithm of Δ_{SC} and Δ_{PG} for a 4LL with OBC (solid line) and PBC (dashed line) for $\varepsilon_0=1$, $\nu=1.2$, and $U=1$. We also indicate the positions of t_{opt}, t^*, and t^{**} for OBC (upper row) and PBC (lower row).

This result can be understood in the following way. In order to obtain pairing instabilities from bare *repulsive* interactions there must be some mechanism which makes the effective (renormalized) interaction attractive in some scattering channel, leading to a Kohn-Luttinger instability. As is well known, in uniform two-dimensional(2D) and 3D systems at weak coupling, the Kohn-Luttinger mechanism leads to extremely small gap scales, but in ladder systems this mechanism is much stronger due to the Fermi-surface nesting and it leads to the opening of a quite large gap in the spin and some charge excitations,[4] which we identify with the ''pseudogap'' Δ_{PG}. Upon increasing the number of chains in the ladders (e.g., from two to four) the nesting becomes weaker, and the magnitude of the gaps is reduced exponentially as the 2D limit is approached. For this reason, starting at $t=0$, Δ_{PG} decreases with increasing t, as can be seen in Fig. 1. On the other hand, the SC phase coherence between the pairs on the different 2LL's, and thus Δ_{SC}, clearly increases with increasing t. Therefore, there is a competition between pseudogap formation, favored by small t, and phase coherence, increasing with t, which leads to an optimal value of $t<1$ for superconductivity, as shown in Fig. 1.

We consider parameters such that there are four bands which cross the Fermi energy, so there are potentially four distinct gapless charge (ρ) and spin (s) modes. Upon bosonization, these modes are represented by the collective bosonic fields, $\phi_{\rho,a}$ and $\phi_{s,a}$, where $a=1$–4 is a band index.[11] Phases (that is to say fixed points of the RG flows) are labeled *CnSm* according[4] to the number of charge (n) and spin (m) modes that remain gapless in the presence of interactions. For the entire range of t, we will show that the 4LL is in the maximally gapped C1S0 phase in which all four spin modes and the three ''relative'' charge modes are gapped. This phase can be considered as the ''SC'' phase of the 4LL. For a range of t, a single, strong-coupling fixed point governs the physics, and hence all the gaps are ''comparable,''[12] this single scale behavior is remeniscent of an anisotropic BCS superconductor. In other ranges, the RG flows pass close to an initial strong-coupling fixed point before finally reaching the C1S0 fixed point. Here, the two gap scales Δ_{PG} and Δ_{SC} are distinct. Specifically, the magnitude of Δ_{PG} is governed by the flow to the initial strong-coupling fixed point, while Δ_{SC} is governed by the second segment of the flow.

We find three regimes of t. For $t^{**}\geqslant t\geqslant t^{*}$, all the gaps are comparable. In contrast, two distinct gap scales are found for $1\geqslant t>t^{**}$ and $t^{*}>t>0$. The two-gap behavior at small t can be understood as follows. For $t=0$, one of the 2LL's has an exponentially larger superconducting gap than the other due to the different Fermi velocities. For finite t, smaller gaps of magnitude Δ_{SC} are induced in the remaining bands by a generalized version of the proximity effect.[6]

a. Inhomogeneous 4LL in weak coupling. We consider a system of two coupled 2LL's described by a Hubbard Hamiltonian with nearest-neighbor hopping along the chains $t_{\parallel}$ (equal to 2 in our units), with interchain hopping $t_{\perp}=1$ along the rungs of *each* 2LL, and with a different hopping $t\leqslant 1$ along the rungs connecting two chains of different 2LL's. For PBC we also include a hopping t connecting the two other chains. In addition, one of the two ladders is shifted in energy by an amount $\varepsilon=\varepsilon_0(1-t^2)$. The electrons interact via a weak, on-site Hubbard interaction $U\ll 2\pi t_{\parallel}$. We fix the number of electrons per site ν to be close to, but not equal to, 1.

TABLE I. Values of the parameters discussed in the text for a 4LL with OBC and PBC, and for $\nu=1.2$ and $\varepsilon_0=1$.

	t^*	t^{**}	t_{min}	α_0	α_1	$\alpha(t=1)$	γ_{AB}	$G_{AB}^{(2)}$
OBC	0.65	0.91	0.81	20.1	16.8	35.1	0.06	0.05
PBC	0.92	1.0	0.91	20.1	38.9	41.8	0.06	0.01

b. First stage renormalization. It is by now a straightforward procedure[4] to derive the weak-coupling RG equations for the large number of distinct running couplings g_i that define the various low-energy two-particle scattering processes. The inputs to the calculation are the initial values $g_i(\tau=0)=\lambda_j U\ll 1$, where the dimensionless constants λ_j depend on the parameters ε_0, t, and ν, and τ is RG flow parameter equal to the ln of the bandwidth (cutoff). Information about the characteristic emergent energy scales is obtained by integrating these equations (typically numerically) until one or more couplings grow to be of order 1.

To be precise, a characteristic gap scale,

$$\Delta\sim\exp[-\tau^{*}]=\exp\left[-\frac{\alpha}{U}+O(\ln U)\right] \tag{1}$$

is obtained from the value of $\tau=\tau^{*}=\alpha/U$ at which some set of couplings diverge. For τ near τ^{*}, there are generally a set of strongly divergent couplings, $g_i\sim G_i(\tau^{*}-\tau)^{-1}$, which all grow to be of order 1 when $(\tau^{*}-\tau)\sim 1$, and so define the new ''strong-coupling fixed point.'' There is also a set of weak or nondivergent couplings, which remain small, when $(\tau^{*}-\tau)\sim 1$.[13,4] What modes are gapped at this strong-coupling fixed point is then determined by bosonizing the model. The order 1 couplings are considered to be large and thus pin (gap) the appropriate modes. As we will discuss below, it may still be necessary to do further analysis to address the fate of the remaining gapless modes at this strong-coupling fixed point, and to account for the effects of the weak residual interactions between them.[6(b)]

Since this analysis straightforwardly extends the previous treatments of the 2LL and 4LL,[4] we will not, here, present the details of the calculation. Some representative results, for parameters listed in the captions, are shown in Fig. 1. For homogeneous $2N$LL's, α increases with the number of chains $2N$, as pointed out in Ref. 4, so that the gaps decrease exponentially with increasing N. In the present case, $\alpha(t)$ must interpolate between the relatively large value for a homogeneous 4LL when $t=1$ and the smaller 2LL value when $t=0$ (see Table I). It turns out that even the dependence of α on t is not monotonic; $\alpha(t)$ first increases with decreasing t until it reaches a maximum value at $t=t_{min}$, and then decreases (i.e., the gaps increase exponentially) as t decreases further.

For $t^{*}<t<t^{**}$ (see Table I) *all* the couplings responsible for gapping the three charge and four spin modes diverge in

proportion to $(\tau^* - \tau)^{-1}$ at $\tau = \tau^*$, so that all gaps at this C1S0 fixed point are "comparable."[12] On the other hand, for $t^* \geqslant t > 0$ or $1 \geqslant t > t^{**}$, only the couplings responsible for the gaps in the spin and relative charge modes within two bands diverge like $(\tau^* - \tau)^{-1}$. More specifically, at the resulting (C3S2) fixed point, $\Delta_{PG} = \Delta$ characterizes the pinning of the spin fields[11] $\phi_{s,1}$, $\phi_{s,2}$ associated with two of the bands (labeled, for simplicity, 1 and 2), and of the relative "superconducting phase," $\theta_{\rho,(1-2)} \equiv (\theta_{\rho,1} - \theta_{\rho,2})/\sqrt{2}$, where $\theta_{\rho,a}$ is the field dual to $\phi_{\rho,a}$. Behavior of this sort has sometimes been interpreted[4,13] as indicating a *transition* to a phase with additional gapless modes, i.e., in this case, from a C1S0 phase for $t^{**} > t > t^*$ to a C3S2 phase for $t < t^*$ and $t > t^{**}$. However, as we shall see (and also Ref. 6) there is a crossover, but no phase transition, at $t = t^*$.

c. Second-stage renormalization. For $t < t^*$, the C3S2 strong-coupling fixed point to which the weak-coupling flows have carried us can be thought of as describing a 1D superconductor in bands $b_A = 1,2$ and two ungapped Luttinger liquids (LL) corresponding to bands $b_B = 3,4$. The fixed-point Hamiltonian thus consists of three gapless charge and two gapless spin modes with a renormalized ultraviolet cutoff Δ_{PG}. However, several residual interactions are left at the end of the first stage of renormalization, and it is necessary to carry out a perturbative stability analysis of the C3S2 fixed point with respect to these interactions.[6] In the weak-coupling limit, it turns out that the four singlet Josephson couplings $(\mathcal{J}_{b_A,b_B})$ between the bands b_A and b_B, i.e., the amplitudes to scatter a zero-momentum pair from b_A to b_B, are the most relevant perturbations and make the C3S2 fixed point unstable. Other interactions are either irrelevant or marginal in the $U \to 0$ limit.

This can be seen by evaluating the scaling dimension (d_J) of the $\mathcal{J}_{b_A,b_B}$, which can be estimated again using bosonization. Specifically, one replaces the gapped fields of the two bands b_A with their expectation values, while all other fields can be taken as nearly free, i.e., their (spin and charge) LL exponents K can be approximately set to 1, since the remaining couplings are small. This procedure (which is standard[6]) leads to the estimate $d_J \approx 1 + 1/(4\,K_{\rho A})$ where $K_{\rho A}$ is the charge Luttinger exponent of the total charge mode of bands b_A. Since[14] $d_J \approx 5/4 < 2$, $\mathcal{J}_{b_A,b_B}$ are relevant perturbations, and diverge at lower energy scales within a strong-coupling RG expansion about the C3S2 fixed point, as anticipated. Again, one can identify the energy scale at which one or more $\mathcal{J}_{b_A,b_B}$ become of order 1 with the freezing of the associated modes and with the opening of corresponding gaps. Specifically, the $\mathcal{J}_{b_A,b_B}$ open a gap in the spin modes (ϕ_{s,b_B} get locked) as well as in the relative charge modes [locking the relative SC phases $(\theta_{\rho,1} + \theta_{\rho,2} - 2\,\theta_{\rho,b_B})/\sqrt{6}$] resulting in a C1S0 phase. Since all $\mathcal{J}_{b_A,b_B}$ are comparable[12] and have the same scaling dimension all these gaps are comparable as well. These smaller gaps are thus identified as Δ_{SC} of the 4LL. In this C1S0 phase, only the global charge mode $\phi_{\rho+}$ remains ungapped. The criterion discussed above yields

$$\Delta_{SC} \approx \Delta_{PG}\, \mathcal{J}_{AB}^{*\,1/(2-d_J)}. \tag{2}$$

Here, $\mathcal{J}^*_{AB}$ is the value of the residual Josephson couplings $\mathcal{J}_{b_A,b_B}$ left at the end of the first stage of renormalization, i.e., when $(\tau^* - \tau) \sim 1$. The point is that $\mathcal{J}^*_{AB}$ is small but *finite* for $t < t^*$.

The optimal t is not given by $t = t^*$. Although the ratio Δ_{SC}/Δ_{PG} is a rapidly decreasing function of decreasing t for $t < t^*$, Δ_{SC} itself still, in general, initially increases due to the exponential increase of Δ_{PG}. Anticipating the fact that the optimum value for t^2 is of the order U, we can determine it by expanding α for small t as $\alpha(t) = \alpha_0 + \alpha_1 t^2 + O(t^4)$ (see Table I). Similarly, we can expand $G_{AB} = G^{(2)}_{AB} t^2 + O(t^4)$. Then from Eq. (2) and from the fact that $\mathcal{J}_{A,B} = \mathcal{J}^{(2)}_{A,B} t^2 + O(t^4)$,

$$\Delta_{SC}(t) \approx \Delta_{PG}(t=0)[\mathcal{J}^{(2)}_{A,B} t^2]^{x_J} e^{-\alpha_1 t^2/U}, \tag{3}$$

where $x_J \equiv 1/(2 - d_J)$. It is now straightforward to determine the value of t which maximizes Δ_{SC},

$$t^2_{opt} = \frac{U}{(2-d_J)\alpha_1} + O(U^2). \tag{4}$$

d. Quantitative considerations. The various quantities describing the t dependence of the gaps discussed here are reported in Table I for $\nu = 1.2$ and $\varepsilon_0 = 1$. The curves in Fig. 1 have been computed as follows. We have taken initial values of the coupling constants corresponding to $U = 1$ and have integrated the RG equations numerically for different values of t until the largest of the couplings (in modulus) has reached the value 1. This determines $\tau^*(t)$, and where needed, $\mathcal{J}^*_{AB}$. The gaps are then evaluated by using Eq. (1) and Eq. (2) with $d_J = 5/4$, respectively.

e. The effects of disorder. Orignac and Giamarchi[9] have shown that disorder is a relevant perturbation (i.e., leads to localization) in the C1S0 phase of the 2LL unless there are strong enough *attractive* interactions such that the Luttinger exponent, $K_{\rho+}$, associated with the gapless total charge mode is greater than 3/2.[14] We now consider the effect of a single-particle disorder potential in the C1S0 phase of the 4LL.

Neglecting forward scattering terms, which turn out not to be crucial here, one can write this potential as

$$H_{dis} = \int d\,x \sum_{b,b'=1}^{4} \hat{v}_{b,b'}(x) e^{i[k_{F,b} + k_{F,b'}]x} + \text{H.c.} \tag{5}$$

Here, the operator $\hat{v}_{b,b'}(x)$ scatters an electron from band b to band b' and from left to right. $\hat{v}_{b,b'}$ is straightforwardly expressed in terms of the bosonic fields. Since in the C1S0 phase all spin fluctuations are gapped, to study the low-energy consequences of weak disorder we can replace all factors of the form $e^{i\alpha\phi_{s,b}}$ by their constant expectation value. The remaining dependence on the charge and dual spin fields is

$$\hat{v}_{b,b'}(x) \sim \frac{\xi_{b,b'}(x)}{2\pi a} e^{i[(\phi_{\rho,b} + \phi_{\rho,b'})/\sqrt{2}]} \cos\left(\frac{\theta_{s,b} - \theta_{s,b'}}{\sqrt{2}} + \eta \right), \tag{6}$$

where the x dependence of the bosonic fields is left implicit, and η is a x-independent phase. The $\xi_{b,b'}(x)$ are the random potentials which, for the sake of definiteness, can be taken to obey a Gaussian distribution $\overline{\xi_{b_1,b_2}(x)\xi_{b_3,b_4}(y)^*}=W_{b_1,b_2|b_3,b_4}D\,\delta(x-y)$, where D is the disorder strength (proportional to the inverse of the mean-free path), and W are numbers of order unity that depend upon details of the band structure.

Now we get to the nub of the problem. The only low-energy fluctuations in the C1S0 phase involve the total charge mode $\phi_{\rho+}$. Since the spin fields are pinned, the dual spin fields are wildly fluctuating, so that any operator that depends on $e^{i\alpha\theta_{s,b}}$ has exponentially falling correlations, and is hence irrelevant in the RG sense. Similarly, since the relative superconducting phases, $\theta_{\rho,b}-\theta_{\rho,b'}$, of the various bands are pinned, any operator with a factor $e^{i\alpha[\phi_{\rho,b}-\phi_{\rho,b'}]}$ is similarly irrelevant. Therefore, both terms in Eq. (6) are irrelevant at low temperatures.

This does not mean disorder is irrelevant. As discussed in Refs. 15 and 16, additional terms get generated in early stages of the RG flow as the gapped fields get integrated out. It is straightforward to show that the most relevant term is obtained only at fourth order in $\hat{v}$, and has the form

$$H_{eff}=\int dx\,\frac{\xi_{eff}(x)}{2\pi a}e^{i\sqrt{8}\phi_{\rho+}}+\text{H.c.},\qquad (7)$$

where ξ_{eff} is an effective disorder potential, which is again Gaussian distributed: $\overline{\xi_{eff}(x)\xi_{eff}(y)^*}=D_{eff}\delta(x-y)$.

For weak enough bare disorder and for energies below the SC gap Δ_{SC}, the RG flow for the renormalized disorder strength D_{eff} associated with Eq. (7) can be easily derived in the usual way.[9] The RG equation reads $d\,D_{eff}/d\,\tau=D_{eff}(3-4K_{\rho+})$, i.e., disorder is irrelevant for $K_{\rho+}>3/4$. This result is very important, as it signals the presence of a localized-delocalized transition in a quasi-one-dimensional system *for purely repulsive interactions*.[14] Moreover, the fact that delocalization is present in weak coupling, i.e., within the range of validity of the present RG procedure, makes the 4LL one of the few low-dimensional models in which one can show the occurrence of delocalized states in a controlled manner. This result is valid provided the disorder does not destroy the gaps, i.e., $D\ll v_F\Delta_{SC}$.

f. Comments and speculations. The present results were derived for small U, where all the emergent energy scales are exponentially small. In many cases, the same physics, and in particular the existence of an optimal degree of inhomogeneity, can be shown[6,17] to apply in the strong-coupling limit, as well, where the gap scales are a significant fraction of the exchange interaction, J. However, quantitative results in this regime generally require some input from numerical experiments.

One can speculate concerning the relevance of the present results for the mechanism of high-temperature superconductivity in two dimensions. Consider an array of coupled ladders as a simple caricature of a 2D stripe ordered superconductor. The present results suggest that T_c may be optimized when the couplings between ladders are less, but not too much less than those within a ladder (i.e., when there is weak but nonvanishing local stripe order) and that an array of inhomogeneous 4LL's (period 4 stripes) is preferable to an array of 2LL's (period 2 stripes). In particular, the superconductivity in coupled 4LL's should be much more robust in the presence of quenched disorder than for the previously studied case[9] of coupled 2LL's. Besides high-T_c superconductors, the effects studied here could be relevant for quasi-one-dimensional systems, such as quantum wires or carbon nanotubes, in which more than one channel crosses the Fermi surface. Finally, this idea could, possibly, give direction to the search for better superconductors.[8]

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