Relativistic action

classical action $S = \int_{t_1}^{t_2} \mathcal{L} \underbrace{dt}_{\gamma d\tau} \equiv \int_{\tau_1}^{\tau_2} L \, d\tau$

fixed x1,x2

$$\Rightarrow \quad L = \gamma \mathcal{L}$$

if we require S to be a Lorentz scalar (LS) , then L is a LS

Construction of L by requiring it to be a LS

1. Free particle $(L_0, \mathcal{L}_0)$

we can use $\quad U^\mu \quad X^\mu$

however, X^μ cannot be used, since we require L to be translation invariant as well

The only LS we can construct is, thus a function of $U_\mu U^\mu = c^2$

$\Rightarrow L_0 = K$ constant !

This means that the principle of minimum action amounts to

minimizing $\int_1^2 d\tau$ which is also called GEODESIC classical action $\mathcal{L}_0 = \frac{K}{\gamma} = K\sqrt{1-\frac{v^2}{c^2}}$

non-relativ. limit $\approx K\left(1-\frac{1}{2}\frac{v^2}{c^2}\right) = K - \frac{K}{2}\frac{v^2}{c^2}$

should be equal to the known action $CONST + \frac{1}{2} m_0 v^2$

$\Rightarrow \quad K = -m_0 c^2$

(2) Introducing interaction with Electromagnetic field

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{EMI}$$

in non-relat. limit $v \to 0$ $\quad \mathcal{L}_{EMI} \approx -U = -q\,\phi$

$\Rightarrow \quad L_{EMI} = \gamma\,\mathcal{L}_{EMI} \approx -\gamma\, q\, \phi$

in general, requiring L to be a LS and using

$$A^\mu = (\phi, \vec{A}) \qquad\qquad U^\mu = \gamma(c, \vec{v})$$

we can "guess" $\quad L_{EMI} = -\frac{q}{c} A^\mu U_\mu$

which for $v \to 0$ yields the correct result $\sim qA_0 = -q\phi$

~~ALTERNATIVE ARGUMENT:~~

~~for $v \to 0$~~

$$\cancel{S_{EM} = \int -q\phi\, dt = -\int e\phi\, \underbrace{dt\, d^3x}_{d^4x}}$$

~~generalisation to $v \neq 0$~~

$$\cancel{-\int j^\mu A_\mu\, d^4x = -\frac{q}{c}\int u^\mu A_\mu\, d\tau}$$

In total

$$L = -m_0 c^2 - \frac{q}{c} A^\mu u_\mu$$

$$\mathcal{L} = L/\gamma = -m_0 c^2 \sqrt{1-\beta^2} - q\phi + \frac{q}{c}\vec{A}\cdot\vec{v}$$

in the non-relat. limit $\simeq -m_0 c^2 + \frac{m_0}{2} v^2 - q\phi + \frac{q}{c}\vec{A}\cdot\vec{v}$

More relativistic invariant treatment

$$S = \int_1^2 L \, dt = \int_1^2 \left(-m_0 c^2 \sqrt{1-\beta^2} - \frac{q}{c} A^\mu \frac{dx_\mu}{dt} \right) dt$$

$$\sqrt{1-\beta^2} = \frac{d\tau}{dt} = \frac{1}{c} \frac{\sqrt{dx^\mu dx_\mu}}{dt}$$

$$\Rightarrow$$

$$S = \int_1^2 \left(-m_0 c \sqrt{dx_\mu dx^\mu} - \frac{q}{c} A^\mu dx_\mu \right)$$

integral over a curve

we can take an arbitrary parametrisation (s) of the curve we are integrating over

$$S = \int_1^2 \left(-m_0 c \sqrt{x'_\mu x'^\mu} - \frac{q}{c} A^\mu x'_\mu \right) ds$$

$$x'_\mu = \frac{dx_\mu}{ds}$$

the lagrangian $\hat{L} = -m_0 c \sqrt{x'_\mu x'^\mu} - \frac{q}{c} A^\mu x'_\mu$

is a LS

Two special types of parametrisation

(A) $s = \tau \Rightarrow \hat{L} = L$

(B) $s = t \Rightarrow \hat{L} = \mathcal{L}$

Equations of motion

as in classical mechanics, the minimum-action principle leads to the Euler-Lagrange equations

$$\frac{d}{ds} \frac{\partial \hat{L}}{\partial x'_\mu} = \frac{\partial \hat{L}}{\partial x_\mu}$$

$$-m_0 c \frac{d}{ds} \frac{x'^\mu}{\sqrt{x'_\nu x'^\nu}} - \frac{q}{c} \frac{d}{ds} A^\mu = -\frac{q}{c} \partial^\mu A^\nu x'_\nu$$

on the trajectory of a particle $\frac{d}{ds} A^\mu = (\partial^\nu A^\mu) x'_\nu$

$$\Rightarrow$$

$$m_0 c \frac{d}{ds} \frac{x'^{\mu}}{\sqrt{x'_\nu x'^\nu}} = \frac{q}{c} x'_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$= \frac{q}{c} x'_\nu F^{\mu\nu}$$

For the two parametrisations:

A $\quad ds = d\tau = \frac{1}{c}\sqrt{dx_\mu \, dx^\mu}$

$$\Rightarrow \sqrt{x'_\nu x'^\nu} = c$$

$$m_0 \frac{du^\mu}{d\tau} = \frac{q}{c} u_\nu F^{\mu\nu}$$

equations for Lorentz force

B $ds = dt$

$$X'^{\mu} = (c, \vec{v}) \Rightarrow \frac{1}{\sqrt{X'^{\nu} X'_{\nu}}} = \frac{\gamma}{c}$$

$$m_0 \frac{d}{dt}\left[(c, \vec{v})\gamma\right]^{\mu} = \frac{q}{c} F^{\mu\nu} \dot{X}_{\nu}$$

SPATIAL COMPONENTS

$$\frac{d}{dt} m_0 \gamma \vec{v} = -\frac{q}{c} F^{i\nu} v_i$$

again Lorentz force

$$S_{EMD} = \underbrace{\int d^4x}_{\int d^3x\,dt} \; \underbrace{\ell_{EMD}(x_\mu)}_{\text{LAGRANGIAN DENSITY: LS}}$$

ℓ_{EMD} QUADRATIC IN $F_{\mu\nu}$, $\mathcal{F}_{\mu\nu}$

(COULD CONTAIN $\frac{d}{dx_\mu}$ ∂_μ, BUT EM. ENERGY DENSITY DOES NOT)

$$F_{\mu\nu}\mathcal{F}^{\mu\nu} = F_{\mu\nu}\,\varepsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta}$$ PSEUDOTENSOR

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu X$$ GAUGE INV.

$$S = \frac{1}{4}\int d^4x\, \overbrace{F_{\mu\nu}F^{\mu\nu}}^{\ell_{EMD}} = \int d^4x\,(E^2 - B^2)\frac{1}{2}$$

$$\propto \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

→ EXTREMUM CONDITION

$$0 \overset{!}{=} \frac{\delta S}{\delta A_\nu} = \partial_\mu \frac{\partial \ell}{\partial(\partial_\mu A_\nu)} + \frac{\partial \ell}{\partial A_\nu}$$

$$\delta \int \ell(\partial_\mu A_\nu)\, d^4x =$$

$$\int \underbrace{\frac{\partial \ell}{\partial(\partial_\mu A_\nu)}\, \delta(\partial_\mu A_\nu)}\; d^4x$$

$$= \int \Bigg(\underbrace{\partial_\mu \left(\frac{\partial \ell}{\partial(\partial_\mu A_\nu)} (\delta A_\nu) \right)}_{\text{TOTAL DERI.} \to 0} - (\delta A_\nu)\, \partial_\mu \frac{\partial \ell}{\partial(\partial_\mu A_\nu)} \Bigg) d^4x$$

$$= - \int d^4x \left(\partial_\mu \frac{\partial \ell}{\partial(\partial_\mu A_\nu)} \right) \delta A_\nu$$

FOR FREE FIELDS

$$\mathcal{L} = \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = F^{\mu\nu}$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = 0$$

$$S_{EMI} = \int -\frac{q}{c} A^\mu U_\mu \, d\tau$$

$$\boxed{\rho \, d^3x = q}$$

$$q U_\mu \, d\tau \rightarrow J_\mu \, d^4x$$ BOTH SCALARS

$$-\int J_\mu A^\mu \, d^4x$$

$$\frac{\partial \mathcal{L}_{EMI}}{\partial A_\nu} = -J^\nu \quad \Rightarrow \partial_\mu F^{\mu\nu} - J^\nu = 0$$

INHOM. MAXWELL EQ.

HOMOG. ALREADY IN $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$